# IMC 2026

## Second Day, July 30, 2026

**Problem 6.**

(a) Is there a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f'(f(x)) = x$$

for every $x \in \mathbb{R}$?

(b) Is there a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f'(f(x)) = |x|$$

for every $x \in \mathbb{R}$?

(10 points)

**Problem 7.** For a continuous function $f\colon [0,1] \to \mathbb{R}$, let $S(f)$ be the union of all straight segments in the plane joining points $(x, 0)$ and $(f(x), 1)$, where $x \in [0,1]$. Let $A(f)$ be the area of $S(f)$. Find the infimum of $A(f)$ over all continuous $f$.

(10 points)

**Problem 8.** Let $n \geq 5$, and suppose that $A = (a_{ij})$ is a real symmetric $n \times n$ matrix such that

$$a_{ii} = 0 \quad \text{and} \quad a_{ij} \in \{-1, 1\} \text{ for } i \neq j.$$

Assume that the scalar products of any two distinct rows of $A$ have the same value. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A$. Prove that

$$\sum_{i=1}^{n} |\lambda_i| \geq 2n - 2$$

and determine all matrices for which equality holds.

(10 points)

**Problem 9.** Let $a_1, a_2, \ldots$ be an infinite sequence of positive real numbers satisfying

$$a_1 + a_2 + \cdots + a_{2n-1} = a_n^2$$

for all positive integers $n$. Prove that $a_n \geq 2n - 1$ for all positive integers $n$.

(10 points)

**Problem 10.** An *infinite chessboard of size* $d > 0$ is obtained by colouring the interiors of the squares of an infinite square grid of side length $d$ alternately white and black following the usual chessboard pattern. The points belonging to the grid lines have neither colour and the grid may be translated and rotated arbitrarily in the plane.

Is it true that for any finite set of points $p_1, \ldots, p_n$ in the plane, there exist $d \in (0, 1)$ and an infinite chessboard of size $d$ such that all the points $p_i$ lie in white squares?

(10 points)