IMC 2026

First Day, July 29, 2026

Solutions

Problem 1. Show that the equation $\cos(\cos x) = \sin(\sin x)$ has no real solutions.

(proposed by Alexander Slávik, Charles University, Prague)

Solution. Assume, for contradiction, that there exists $x \in \mathbb{R}$ such that $\cos(\cos x) = \sin(\sin x)$. Using $\cos t = \sin\left(\frac{\pi}{2} - t\right)$, we may rewrite this as

$$\sin\left(\frac{\pi}{2} - \cos x\right) = \sin(\sin x).$$

Set

$$A = \frac{\pi}{2} - \cos x \qquad \text{and} \qquad B = \sin x.$$

Since $\sin x, \cos x \in [-1, 1]$, we have

$$A \in \left[\frac{\pi}{2} - 1, \frac{\pi}{2} + 1\right] \qquad \text{and} \qquad B \in [-1, 1].$$

The general solutions of $\sin A = \sin B$ are

$$A - B = 2k\pi \qquad \text{or} \qquad A + B = (2k+1)\pi, \qquad k \in \mathbb{Z}.$$

The above bounds on A and B force $k = 0$. Hence either

$$\frac{\pi}{2} - \cos x = \sin x$$

or

$$\frac{\pi}{2} - \cos x = \pi - \sin x.$$

After rearranging, these equations become, respectively,

$$\sin x + \cos x = \frac{\pi}{2}$$

and

$$\sin x - \cos x = \frac{\pi}{2}.$$

However,

$$\sin x \pm \cos x \leq \sqrt{2},$$

as follows, for example, from

$$\sin x \pm \cos x = \sqrt{2}\,\sin\left(x \pm \frac{\pi}{4}\right).$$

Since

$$\sqrt{2} < \frac{\pi}{2},$$

neither equality is possible. Hence the equation

$$\cos(\cos x) = \sin(\sin x)$$

has no real solutions.

Problem 2. Let n be a positive integer. Suppose that A and B are $n \times n$ matrices with real entries such that

$$A^\top A + BB^\top = AB + BA,$$

where $X^\top$ denotes the transpose of matrix X.

Does this imply that $AB = BA$?

(proposed by Nikolaos Kolliopoulos, University of Cyprus)

Solution 1. Let $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^n$. By taking traces of both sides,

$$0 = \operatorname{tr}(\mathrm{A^\top A + BB^\top - AB - BA}) = \operatorname{tr}(\mathrm{A^\top A}) + \operatorname{tr}(\mathrm{BB^\top}) - \operatorname{tr}(\mathrm{AB}) - \operatorname{tr}(\mathrm{BA})$$

$$= \sum_{i,j=1}^n a_{ij}^2 + \sum_{i,j=1}^n b_{ji}^2 - 2\sum_{i,j=1}^n a_{ij}b_{ji} = \sum_{i,j=1}^n (a_{ij} - b_{ji})^2.$$

Hence, $a_{ij} = b_{ji}$ for every i, j, so $B = A^\top$.

By substituting $B = A^\top$ back into the condition, we get

$$2BA = A^\top A + BB^\top = AB + BA,$$

$$BA = AB,$$

so the answer is YES.

Remark. If $B = A^\top$ and $AB = BA$ then $A^\top A = A^\top A$,so A must be a normal matrix. Then the condition is satisfied as

$$A^\top A + BB^\top = AB + BA = 2AB.$$

Solution 2. Observe that the desired equality can be written as

$$(A - B^\top)^\top (A - B^\top) = AB - A^\top B^\top$$

where the left-hand side is a symmetric and non-negative definite matrix over $\mathbb{R}$, meaning that it has only real non-negative eigenvalues. On the other hand, for the right-hand side we can compute

$$\operatorname{tr}(\mathrm{AB - A^\top B^\top}) = \operatorname{tr}(\mathrm{AB}) - \operatorname{tr}(\mathrm{A^\top B^\top}) = \operatorname{tr}(\mathrm{AB}) - \operatorname{tr}(\mathrm{BA})^\top = 0.$$

Hence, the matrix $(A - B^\top)^\top (A - B^\top) = AB - A^\top B^\top$ must have all its non-negative eigenvalues equal to 0, and because it is a symmetric matrix with real entries and thus a diagonalizable matrix, it has to be the zero matrix. This means that $A - B^\top = 0$ so that $A = B^\top$. Then, the solution can be completed like in the first solution.

Problem 3. Consider a deck of $n \geq 2$ cards labeled $1, 2, \ldots, n$. An *alternating shuffle* of the deck is performed as follows. We split the deck into two non-empty stacks. We then sort the first stack in increasing order, and the second stack in decreasing order. Finally, we alternately take cards from the first and second stacks (starting with the first). If one of the stacks runs out, the remaining cards from the other stack are placed at the end. How many different final orders of the deck can be obtained in this way?

Example: Suppose $n = 6$, the first stack is $A = (1, 3)$, and the second stack is $B = (6, 5, 4, 2)$. Then the order resulting from the alternating shuffle is $(1, 6, 3, 5, 4, 2)$.

(proposed by Daniel Volostnov, Neapolis University Paphos, Cyprus)

Solution. **Setup.** Define the *reference string* $\mathcal{S} = ABAB\cdots$ of length n. For a given subset Y of cards in the first stack, let $\mathcal{S}_P \in \{A, B\}^n$ be the *source string* of $P(Y)$: $(\mathcal{S}_P)_i = A$ if the card at position i of P came from deck A, and $(\mathcal{S}_P)_i = B$ otherwise.

For $i \in \{1, 2, \ldots, n\}$, define

$$B_i = \left\{ P(Y) \;\middle|\; \text{the smallest index } j \text{ with } (\mathcal{S}_P)_j \neq \mathcal{S}_j \text{ equals } i \right\},$$

and let $B_{n+1} = \left\{ P(Y) \;\middle|\; \mathcal{S}_P = \mathcal{S} \right\}$ (i.e. the source string matches the reference string perfectly).

For each i, let G_i be the directed graph on vertex set $\{1, 2, \ldots, n\}$ where there is an edge from x to y whenever $P_x > P_y$, encoding the relative-order structure common to all orders of B_i.

Throughout, assume $i < j$.

Step 1. *$B_i \cap B_j = \varnothing$ whenever $j - i$ is odd.*

Consider the graphs G_i and G_j, and look at vertices $n-1$ and n. For i and j of different parity, the edge between $n-1$ and n is directed in opposite ways in G_i and G_j. If a common order existed in $B_i \cap B_j$, it would simultaneously require $P_{n-1} > P_n$ and $P_{n-1} < P_n$, a contradiction.

Step 2. *$B_i \cap B_j = \varnothing$ whenever $j - i$ is even and $j - i > 2$.*

Consider the graphs G_i and G_j at vertices i and $i + 2$. By the same argument as in Step 1, the edges between these vertices point in opposite directions, giving a contradiction.

Step 3. *B_{n+1} creates a single exceptional overlap with B_n.*

The set B_{n+1} is on equal footing with the other B_i: Steps 1 and 2 apply to it as well, so B_{n+1} is disjoint from all B_i except possibly B_{n-1} and B_n (by parity). In fact, B_{n+1} and B_n share exactly one order: the sequence $P = (1, n, 2, n-1, \ldots)$ produced by $A = \{1, 2, \ldots, \lceil n/2 \rceil\}$ and $B = \{n, n-1, \ldots, \lfloor n/2 \rfloor + 1\}$. This is because the edge $n-1 \to n$ does not appear in G_{n+1}, so the graph constraint of G_n is also satisfied. No other order of B_{n+1} lies in B_n.

From Steps 1–3 it follows that $B_1 \cup B_2 \cup \cdots \cup B_{n+1} = \{P(Y) : Y \subseteq S\}$, and the only non-empty pairwise intersections are $D_i := B_i \cap B_{i+2}$ for $i = 1, \ldots, n-1$, together with the single exceptional order $|B_n \cap B_{n+1}| = 1$.

Step 4. *Computing $|B_i|$.*

The graph G_i consists of two directed paths ("bamboo sticks"). Since each B_i encodes all possible subsets Y whose source string first deviates from $\mathcal{S}$ at position i, the bamboo lengths depend on the parity of i:

- if i is odd: $|B_i| = \dbinom{n}{(i-1)/2}$,
- if i is even: $|B_i| = \dbinom{n}{n - i/2 + 1}$.

Since all B_i together (for $i = 1, \ldots, n+1$) partition the set of source-string encodings of all 2^n subsets, we have

$$\sum_{i=1}^{n+1} |B_i| = 2^n.$$

Step 5. *Computing the overlaps* $|D_i|$ *where* $D_i = B_i \cap B_{i+2}$.

From Steps 1 and 2, $D_i \cap D_j = \varnothing$ for $i \neq j$. Consider the union graph G' of G_i and G_{i+2}. For odd i, the graph G' is almost identical to G_{i+2} except that edges $i \to i+1$ and $i+1 \to i+2$ are added, which fix the relative order of $(i+1)/2$ vertices. Removing these determined vertices leaves two bamboo sticks, one with $(i-1)/2$ vertices. Hence for odd i,

$$|D_i| = \binom{n-(i+1)/2}{(i-1)/2}.$$

Substituting $i = 2h+1$ gives $|D_i| = \binom{n-1-h}{h}$.

An analogous argument for even $i = 2h+2$ yields $|D_i| = \binom{n-2-h}{h}$.

Step 6. *Assembling the answer.*

By inclusion–exclusion,

$$N(n) = \sum_{i=1}^{n+1} |B_i| - 1 - \sum_{i=1}^{n-1} |D_i|,$$

where -1 accounts for the single exceptional overlap $|B_n \cap B_{n+1}| = 1$ from Step 3.

The sum of $|B_i|$ equals 2^n (Step 4).

The sum over odd-indexed D_i $(i = 1, 3, 5, \ldots)$: when n is odd this equals $F_n - 1$; when n is even this equals F_n. Similarly, the sum over even-indexed D_i $(i = 2, 4, 6, \ldots)$: when n is odd this equals F_{n-1}; when n is even this equals $F_{n-1} - 1$.

In both cases the total is

$$\sum_{i=1}^{n-1} |D_i| = F_n + F_{n-1} - 1 = F_{n+1} - 1.$$

Therefore,

$$\boxed{N(n) = 2^n - 1 - (F_{n+1} - 1) = 2^n - F_{n+1}},$$

where $F_{n+1} = F_n + F_{n-1}$ is the $(n{+}1)$-th Fibonacci number (with $F_1 = F_2 = 1$).

Problem 4. Let $x_1 > 0$. Define the sequence $\{x_n\}$ by the recurrence

$$x_{n+1} = \arctan\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \quad \text{for all } n \geq 1.$$

Find $\lim_{n\to\infty} x_n\sqrt{\ln n}$, where $\ln x$ denotes the natural logarithm of x.

(proposed by Wanlong Han, Henan, China)

Solution. Let $y_n = \frac{x_1+x_2+\cdots+x_n}{n}$. Then we have the recurrence relation:

$$y_{n+1} = \frac{ny_n + x_{n+1}}{n+1}. \tag{1}$$

where $x_{n+1} = \arctan y_n$. By mathematical induction, $x_n > 0$ holds for all positive integers n.

Since $\arctan t < t$ for all $t > 0$, it follows that $x_{n+1} = \arctan y_n < y_n$. Substituting into equation (1):

$$y_{n+1} < \frac{ny_n + y_n}{n+1} = \frac{(n+1)y_n}{n+1} = y_n,$$

which implies $\{y_n\}$ is strictly decreasing. As $y_n > 0$ for all n, by the Monotone Convergence Theorem, $\{y_n\}$ converges. Let $\lim_{n\to\infty} y_n = y$.

Furthermore, since $y_n > y_{n+1}$, we have

$$x_{n+1} = \arctan y_n > \arctan y_{n+1} = x_{n+2},$$

so $\{x_n\}$ is also strictly decreasing and bounded below by 0. By the Monotone Convergence Theorem, $\{x_n\}$ converges. Let $\lim_{n\to\infty} x_n = x$.

Taking limits on both sides of the recurrence:

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \arctan\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) = \arctan y.$$

It is obvious that $y = x$. By the preservation of inequalities under limits, $x \geq 0$. If $x > 0$, then

$$x = \arctan y = \arctan x < x,$$

which is a contradiction. Therefore $x = 0$, and hence $y = 0$.

Note that

$$\frac{x_n}{y_n} = \frac{\arctan y_n}{y_n} \to 1 \text{ as } n \to \infty,$$

so x_n and y_n are equivalent infinitesimals as $n \to \infty$.

By Stolz's Theorem (∞/∞ form):

$$\lim_{n\to\infty} \frac{1/x_n^2}{\ln n} = \lim_{n\to\infty} \frac{1/y_n^2}{\ln n} = \lim_{n\to\infty} \frac{\frac{1}{y_{n+1}^2} - \frac{1}{y_n^2}}{\ln(n+1) - \ln n}.$$

Simplify the numerator:

$$\frac{1}{y_{n+1}^2} - \frac{1}{y_n^2} = \frac{y_n^2 - y_{n+1}^2}{y_n^2 y_{n+1}^2} = \frac{(y_n - y_{n+1})(y_n + y_{n+1})}{y_n^2 y_{n+1}^2}.$$

As $n \to \infty$, $y_{n+1} \sim y_n$, so $y_n + y_{n+1} \sim 2y_n$ and $y_n^2 y_{n+1}^2 \sim y_n^4$.

For the denominator:

$$\ln(n+1) - \ln n = \ln\left(1 + \frac{1}{n}\right) \sim \frac{1}{n} \quad (n \to \infty).$$

Thus

$$\lim_{n\to\infty}\frac{\frac{1}{y_{n+1}^2}-\frac{1}{y_n^2}}{\ln(n+1)-\ln n}=\lim_{n\to\infty}\frac{(y_n-y_{n+1})\cdot 2y_n}{y_n^4}\cdot n=2\lim_{n\to\infty}\frac{n\left(y_n-y_{n+1}\right)}{y_n^3}.$$

Now compute y_n-y_{n+1}:

$$y_n-y_{n+1}=y_n-\frac{ny_n+x_{n+1}}{n+1}=\frac{(n+1)y_n-ny_n-x_{n+1}}{n+1}=\frac{y_n-x_{n+1}}{n+1}=\frac{y_n-\arctan y_n}{n+1}.$$

Substitute back into the limit:

$$2\lim_{n\to\infty}\frac{n\cdot\frac{y_n-\arctan y_n}{n+1}}{y_n^3}=2\lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{y_n-\arctan y_n}{y_n^3}.$$

Since $\frac{n}{n+1}\to 1$ as $n\to\infty$, and using the Taylor expansion $t-\arctan t\sim\frac{t^3}{3}$ for $t\to 0^+$:

$$2\lim_{t\to 0^+}\frac{t-\arctan t}{t^3}=2\cdot\frac{1}{3}=\frac{2}{3}.$$

Therefore

$$\lim_{n\to\infty}\frac{1/x_n^2}{\ln n}=\frac{2}{3}$$

and we conclude

$$\lim_{n\to\infty}x_n\sqrt{\ln n}=\sqrt{\frac{3}{2}}=\frac{\sqrt{6}}{2}$$

Problem 5. Prove that there exists a constant $C>0$ such that for every pair A,B of positive integers, there is a real polynomial $p(x)$ with

$$p(0)^2 > \sum_{i=1}^{A} p(-i)^2 + \sum_{i=1}^{B} p(i)^2 \quad \text{and} \quad \deg p < C\sqrt{AB}.$$

(proposed by Géza Kós, Loránd Eötvös University, Budapest)

Solution. Without loss of generality we can assume $A \ge B$.

Let k be an odd integer with $7\sqrt{AB} \le k \le 7\sqrt{AB}+2$, and consider the Chebishev polynomial $T_k(x)$. Let $\omega = \arccos \dfrac{A-B}{A+B}$, and let $u_0 = \cos\omega_0$ be the greatest root of $T_k(x)$ in the interval $\left[-1, \dfrac{A-B}{A+B}\right]$. Since k is odd, we have $T_k(0)=0$, so $u_0 \ge 0$. Moreover, $\omega \le \omega_0 < \omega + \dfrac{\pi}{k}$.

The requested polynomial will be constructed as

$$p(x) = \frac{T_k\left(u_0 + \frac{1+u_0}{A}x\right)}{x}.$$

Notice that for all $x \in [-A,B]$ we have

$$u_0 + \frac{1+u_0}{A}x \ge u_0 + \frac{1+u_0}{A}(-A) = -1 \quad \text{and} \quad u_0 + \frac{1+u_0}{A}x \le \frac{A-B}{A+B} + \frac{1+\frac{A-B}{A+B}}{A}B = 1.$$

Hence, for $x \in [-A,B]$ we have $\left|T_k\left(u_0 + \frac{1+u_0}{A}x\right)\right| \le 1$ and $|p(x)| \le \dfrac{1}{|x|}$, and therefore

$$\sum_{i=1}^{A} |p(-i)|^2 + \sum_{i=1}^{B} |p(i)|^2 < 2\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{3} < 4.$$

In order to estimate $p(0)$, notice that

$$|p(0)| = \frac{1+u_0}{A}|T_k'(u_0)| \ge \frac{1}{A}|T_k'(u_0)|.$$

From $\cos(kt) = T_k(\cos t)$ we get

$$-k\sin(kt) = T_k'(\cos t)\cdot(-\sin t)$$

$$T'(u_0) = T'(\cos\omega_0) = k\frac{\sin(k\omega_0)}{\sin\omega_0} = \pm\frac{k}{\sin\omega_0}.$$

Since $\omega_0 \le \min\left(\omega + \frac{\pi}{k}, \frac{\pi}{2}\right)$,

$$\sin\omega_0 < \sin\omega + \frac{\pi}{k} \le \sqrt{1-\cos^2\omega} + \frac{\pi}{7\sqrt{AB}}$$

$$= \sqrt{1-\left(\frac{A-B}{A+B}\right)^2} + \frac{\pi/7}{\sqrt{AB}} = \frac{2\sqrt{AB}}{A+B} + \frac{\pi/7}{\sqrt{AB}} < 3\sqrt{\frac{B}{A}}.$$

Hence,

$$|p(0)| \ge \frac{1}{A}|T_k'(u_0)| = \frac{1}{A}\cdot\frac{k}{\sin\omega_0} > \frac{7\sqrt{AB}}{A\cdot 3\sqrt{\frac{B}{A}}} > 2,$$

so indeed

$$|p(0)|^2 > 4 > \sum_{i=1}^{A} |p(-i)|^2 + \sum_{i=1}^{B} |p(i)|^2.$$

The degree of p is

$$\deg p = k-1 < 7\sqrt{AB}+1 \le 8\sqrt{AB}.$$

So, $C=8$ is suitable.