IMC 2025

Second Day, July 31, 2025

Solutions

Problem 6. Let $f\colon (0,\infty) \to \mathbb{R}$ be a continuously differentiable function, and let $b > a > 0$ be real numbers such that $f(a) = f(b) = k$. Prove that there exists a point $\xi \in (a,b)$ such that

$$f(\xi) - \xi f'(\xi) = k.$$

(proposed by Alberto Cagnetta, Università degli Studi di Udine)

Solution. Observe that if we consider $g(x) = f(x)/x$ and take its derivative, we get

$$g'(x) = \frac{xf'(x) - f(x)}{x^2},$$

where the numerator is almost the expression we have in the problem.

Now we apply Cauchy's theorem to the functions $g(x) = f(x)/x$ and $h(x) = 1/x$, well-defined over $[a,b]$. This gives us the existence of a value $\xi \in [a,b]$ such that

$$\frac{g(a) - g(b)}{h(a) - h(b)} = \frac{g'(\xi)}{h'(\xi)}.$$

Here,

$$\frac{g'(\xi)}{h'(\xi)} = \frac{\frac{\xi f'(\xi) - f(\xi)}{\xi^2}}{-\frac{1}{\xi^2}} = f(\xi) - \xi f'(\xi)$$

and

$$\frac{g(a) - g(b)}{h(a) - h(b)} = \frac{\frac{f(a)}{a} - \frac{f(b)}{b}}{\frac{1}{a} - \frac{1}{b}} = k,$$

which concludes the proof.

Problem 7. Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all nonempty subsets $M \subseteq \mathbb{Z}_{>0}$ satisfying both of the following properties:

(a) if $x \in M$, then $2x \in M$,

(b) if $x, y \in M$ and $x + y$ is even, then $\dfrac{x+y}{2} \in M$.

(proposed by Alexandr Bolbot, Novosibirsk State University)

Solution. Note that M is closed under addition since $x + y = \frac{2x+2y}{2}$. Therefore it is closed under multiplication by an arbitrary natural number. Also, M contains some odd numbers since $x \in M \implies \frac{x+2x}{2} = \frac{3x}{2} \in M$, and we can repeat this until all factors of 2 are removed from a number.

Let d be the greatest common divisor of all members from M. Then $M \subseteq d\,\mathbb{Z}_{>0}$ and d is odd. Since d can be represented in the form

$$d = v_1a_1 + v_2a_2 + \ldots + v_ka_k - v_{k+1}a_{k+1} - v_{k+2}a_{k+2} - \ldots - v_na_n$$

for some $a_i \in M$ and $v_i \in \mathbb{Z}_{>0}$, there exist two members of M with difference d.

Let c be the minimal element of M, $c < a$, and $a, a + d \in M$. We choose the largest $x \in M$ such that $x < a$. Then the only elements of M in the interval $[x, a + d]$ are x, a, and $a + d$, since $M \subseteq d\,\mathbb{Z}$. Meanwhile, $x < \frac{x+a}{2} < a$, so $x + a$ must be odd, hence $x + a + d$ is even, and then $x < \frac{x+a+d}{2} < a + d$ means $x = a - d$. Thus, the following implication holds:

$$(a, a + d \in M \text{ and } c < a) \implies a - d \in M.$$

Similarly, by setting $x \in M$ to be the smallest such that $x > a$ (we know that such an element exists, since $2a \in M$), we obtain

$$a - d, a \in M \implies a + d \in M.$$

We thus get that M is obtained as the set of elements of the arithmetic progression $c + kd$ ($k \in \mathbb{Z}_{\geq 0}$). Obviously, this set satisfies both of the properties (a) and (b). Hence, we have proved that M satisfies these properties if and only if $M = \{nd \mid m \leqslant n \in \mathbb{N}\}$ for some $m \in \mathbb{N}$ and odd d.

Problem 8. For an $n \times n$ real matrix $A \in M_n(\mathbb{R})$, denote by A^{R} its counter-clockwise 90° rotation. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{\mathsf{R}} = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 8 \\ 1 & 4 & 7 \end{bmatrix}.$$

Prove that if $A = A^{\mathsf{R}}$ then for any eigenvalue λ of A, we have $\operatorname{Re}\lambda = 0$ or $\operatorname{Im}\lambda = 0$.

(proposed by Jan Kuś, University of Warwick)

Solution. If $\lambda = 0$, the claim holds as $0 \in \mathbb{R}$. Assume $\lambda \neq 0$ is an eigenvalue of A with a corresponding eigenvector $x \in \mathbb{C}^n \setminus \{0\}$.

We will first express the operation $A \mapsto A^{\mathsf{R}}$ algebraically. The element at position (i, j) in A ends up at position $(n+1-j, i)$ in A^{R}. Thus, the rotation is defined by the relation $(A^{\mathsf{R}})_{i,j} = A_{j,n+1-i}$.

Let J be the matrix where $J_{i,j} = \delta_i^{n+1-j}$. The operation of transposing A and then reversing the rows gives the matrix $JA^\top$. The (i, j)-th element of this matrix is

$$(JA^\top)_{i,j} = \sum_{k=1}^{n} J_{i,k}(A^\top)_{k,j} = (A^\top)_{n+1-i,j} = A_{j,n+1-i}.$$

This matches the definition of A^{R}, so we get the identity $A^{\mathsf{R}} = JA^\top$. Note that the matrix J is symmetric ($J = J^\top$) and it is its own inverse ($J^2 = I$).

The given condition $A = A^{\mathsf{R}}$ thus means $A = JA^\top$. Left-multiplying by J yields

$$JA = J(JA^\top) = (J^2)A^\top = A^\top. \tag{$*$}$$

Now, consider the standard Hermitian inner product $(u, v) = v^* u$ on $\mathbb{C}^n$. We evaluate (Ax, Ax) in two ways. First, using our choice of x as an eigenvector corresponding to λ:

$$(Ax, Ax) = (\lambda x, \lambda x) = |\lambda|^2 \|x\|^2.$$

Second, using the adjoint property and ($*$):

$$(Ax, Ax) = (A^* Ax, x) = (A^\top Ax, x) = (JA(\lambda x), x) = \lambda(JAx, x) = \lambda^2(Jx, x).$$

Together, these give us $|\lambda|^2\|x\|^2 = \lambda^2(Jx, x)$.

The term (Jx, x) is real, since $(Jx, x)^* = (J^* x, x) = (Jx, x)$ because J is real and symmetric. Since $\lambda \neq 0$ and $x \neq 0$, the left side $|\lambda|^2\|x\|^2$ is a positive real number. This implies that $\lambda^2(x, Jx)$ must also be a positive real number. And as (x, Jx) is real, so is λ^2.

Thus, either λ is real (if $\lambda^2 > 0$) or its real part is 0 (if $\lambda^2 < 0$). This completes the proof.

Problem 9. Let n be a positive integer. Consider the following random process which produces a sequence of n distinct positive integers $X_1, X_2, \ldots, X_n$.
First, X_1 is chosen randomly with $\mathbb{P}(X_1 = i) = 2^{-i}$ for every positive integer i. For $1 \le j \le n-1$, having chosen $X_1, \ldots, X_j$, arrange the remaining positive integers in increasing order as $n_1 < n_2 < \cdots$, and choose X_{j+1} randomly with $\mathbb{P}(X_{j+1} = n_i) = 2^{-i}$ for every positive integer i.
Let $Y_n = \max\{X_1, \ldots, X_n\}$. Show that

$$\mathbb{E}[Y_n] = \sum_{i=1}^{n} \frac{2^i}{2^i - 1}$$

where $\mathbb{E}[Y_n]$ is the expected value of Y_n.

(proposed by Jan Kuś and Jun Yan, University of Warwick)

Solution 1. For each $j \in [n] = \{1, 2, \ldots, n\}$, let $Y_j = \max\{X_1, \ldots, X_j\}$. We use induction on j to show that

$$\mathbb{E}[Y_j] = \sum_{i=1}^{j} \frac{2^i}{2^i - 1} \quad \text{for all } j \in [n].$$

The base case $j = 1$ follows easily from definition.

For the inductive step, it suffices to show that $\mathbb{E}[Y_{j+1} - Y_j] = \frac{2^{j+1}}{2^{j+1}-1}$ for every $j \in [n-1]$. Note that $Y_{j+1} \neq Y_j$ if and only if $X_{j+1} > Y_j = \max\{X_1, \ldots, X_j\}$, in which case $Y_{j+1} = X_{j+1}$. Thus,

$$\mathbb{E}[Y_{j+1} - Y_j] = \mathbb{P}[X_{j+1} > Y_j] \cdot \mathbb{E}[X_{j+1} - Y_j \mid X_{j+1} > Y_j].$$

To compute $\mathbb{P}[X_{j+1} > Y_j]$, note that for any fixed pairwise distinct positive integers $a_1, \ldots, a_j$ and $a > \max\{a_1, \ldots, a_j\}$,

$$\begin{aligned}\mathbb{P}[(X_1, \ldots, X_{j+1}) = (a, a_1, \ldots, a_j)] &= \mathbb{P}[(X_1, \ldots, X_{j+1}) = (a_1, a, \ldots, a_j)]/2 \\ &= \mathbb{P}[(X_1, \ldots, X_{j+1}) = (a_1, a_2, a, \ldots, a_j)]/4 \\ &= \cdots = \mathbb{P}[(X_1, \ldots, X_{j+1}) = (a_1, \ldots, a_j, a)]/2^j.\end{aligned}$$

Therefore, summing over all possible $a_1, \ldots, a_j$ and $a > \max\{a_1, \ldots, a_j\}$, we see that

$$\mathbb{P}[X_{j+1} > Y_j] = \frac{2^j}{\sum_{i=0}^{j} 2^i} = \frac{2^j}{2^{j+1} - 1}.$$

To finish, it is easy to see that

$$\mathbb{E}[X_{j+1} - Y_j \mid X_{j+1} > Y_j] = \sum_{t=1}^{\infty} t \cdot \mathbb{P}[X_{j+1} = Y_j + t \mid X_{j+1} > Y_j] = \sum_{t=1}^{\infty} \frac{t}{2^t} = 2.$$

Solution 2. Since Y_n takes values in $\mathbb{Z}_{>0}$,

$$\mathbb{E}[Y_n] = \sum_{k=1}^{\infty} \mathbb{P}[Y_n \ge k].$$

For each $k \in [n]$, $\mathbb{P}[Y_n \ge k] = 1$, while for each $k > n$,

$$\mathbb{P}[Y_n \ge k] = 1 - \mathbb{P}[Y_n < k] = 1 - \mathbb{P}[X_1, \ldots, X_n < k] = 1 - \prod_{i=1}^{n} \left(1 - \frac{1}{2^{k-i}}\right).$$

Note that this formula also works for every $k \in [n]$, as the $i = k$ term in the product is 0. Thus, it suffices to show that

$$\sum_{i=1}^{n} \frac{2^i}{2^i - 1} = \sum_{k=1}^{\infty} \left(1 - \prod_{i=1}^{n} \left(1 - \frac{1}{2^{k-i}}\right)\right).$$

The case of $n = 1$ is easy to verify. Using induction on n, it suffices to show that for every $n \geq 2$,

$$\frac{1}{1 - \frac{1}{2^n}} = \sum_{k=1}^{\infty} \left(\prod_{i=1}^{n-1} \left(1 - \frac{1}{2^{k-i}}\right) - \prod_{i=1}^{n} \left(1 - \frac{1}{2^{k-i}}\right)\right) = \sum_{k=1}^{\infty} \left(\frac{1}{2^{k-n}} \prod_{i=1}^{n-1} \left(1 - \frac{1}{2^{k-i}}\right)\right).$$

Indeed, for every $N > n$, after multiplying by $1 - \frac{1}{2^n}$, the sum on the right telescopes as

$$\left(1 - \frac{1}{2^n}\right) \sum_{k=1}^{N} \left(\frac{1}{2^{k-n}} \prod_{i=1}^{n-1} \left(1 - \frac{1}{2^{k-i}}\right)\right) = \sum_{k=1}^{N} \left(\left(\frac{1}{2^{k-n}} - \frac{1}{2^k}\right) \prod_{i=1}^{n-1} \left(1 - \frac{1}{2^{k-i}}\right)\right) =$$

$$= \sum_{k=1}^{N} \left(\prod_{i=1}^{n} \left(1 - \frac{1}{2^{k+1-i}}\right) - \prod_{i=1}^{n} \left(1 - \frac{1}{2^{k-i}}\right)\right) = \prod_{i=1}^{n} \left(1 - \frac{1}{2^{N+1-i}}\right).$$

Taking N to infinity finishes the proof.

Solution 3 (sketch). It can be shown by induction or another method that for any sequence of positive integers $a_1 < a_2 < \cdots < a_n$,

$$\mathbb{P}[\{X_1, \ldots, X_n\} = \{a_1, \ldots, a_n\}] = 2^{-\sum_{i=1}^{n} a_i} \prod_{i=1}^{n} (2^i - 1).$$

For any $a_1 < a_2 < \cdots < a_n$, let $d_1 = a_1$ and $d_{i+1} = a_{i+1} - a_i$ for $i \in [n-1]$, so

$$\mathbb{P}[\{X_1, \ldots, X_n\} = \{d_1, d_1 + d_2, \ldots, d_1 + \cdots + d_n\}] = 2^{-\sum_{i=1}^{n} (n+1-i) d_i} \prod_{i=1}^{n} (2^i - 1).$$

Note that $(a_1, \ldots, a_n) \mapsto (d_1, \ldots, d_n)$ is a bijection between strictly increasing sequences in $\mathbb{Z}_{>0}$ of length n and $(\mathbb{Z}_{>0})^n$, so, using $\sum_{i \geq 1} x^i = x/(1-x)$ and $\sum_{i \geq 1} i x^i = x/(1-x)^2$, we get

$$\begin{aligned}
\mathbb{E}[Y_n] &= \prod_{i=1}^{n} (2^i - 1) \sum_{d_1, \ldots, d_n \geq 1} \left(\sum_{i=1}^{n} d_i\right) 2^{-\sum_{j=1}^{n} (n+1-j) d_j} \\
&= \prod_{i=1}^{n} (2^i - 1) \sum_{i=1}^{n} \left(\sum_{d_i \geq 1} d_i 2^{-(n+1-i) d_i}\right) \prod_{j \neq i} \left(\sum_{d_j \geq 1} 2^{-(n+1-j) d_j}\right) = \cdots = \sum_{i=1}^{n} \frac{2^i}{2^i - 1}.
\end{aligned}$$

Problem 10. For any positive integer N, let S_N be the number of pairs of integers $1 \le a, b \le N$ such that the number $(a^2+a)(b^2+b)$ is a perfect square. Prove that the limit

$$\lim_{N\to\infty} \frac{S_N}{N}$$

exists and find its value.

(proposed by Besfort Shala, University of Bristol)

Solution. Throughout the solution, we use the Vinogradov notation $A \ll B$ to mean $A = O(B)$, which in turn means that there exists a constant $C > 0$, independent of the quantities A and B, such that $|A| \le C|B|$, on the entirety of the domain where A and B are defined (for us, this will always be the interval $[1, \infty)$.)

We will show that the limit equals 1, corresponding to the trivial solutions $a = b$. Note that $(a^2+a)(b^2+b)$ is a perfect square if and only if $a^2+a = dz_1^2$ and $b^2+b = dz_2^2$ for some square-free d and $z_1, z_2 \in \mathbb{Z}_{>0}$. From this point on, all sums over d will be over square-free positive integers. Multiplying the equations by 4 and setting $y_i = 2z_i$, we get

$$S_N = \sum_{d \ll N^2} c_d(N)^2 + O(1),$$

where $c_d(N)$ is the number of solutions to $(2k+1)^2 - dy^2 = 1$ with $1 \le k \le N$ and $1 \le y \le N/2$ with y even. Other than for the purpose of identifying the trivial solutions, we will work with Pell's equation $x^2 - dy^2 = 1$ with $1 \le x, y \ll N$. Split the sum as

$$\sum_{\substack{d \ll N^2 \\ c_d(N) \le 1}} c_d(N) + \sum_{\substack{d \ll N^2 \\ c_d(N) > 1}} c_d(N)^2.$$

Note that if $d \gg N$, then the size of the second solution $x_2 = x_1^2 + dy_1^2$ (coming from $x_2 + y_2\sqrt{d} = (x_1 + y_1\sqrt{d})^2$, where $x_1 + y_1\sqrt{d}$ is the fundamental solution) is $\gg d \gg N$. Hence we may assume that $d \ll N$ if $c_d(N) > 1$ (with a suitable choice of hidden constants). Denote the second sum by E (for error, which we will bound momentarily). The first sum is easily manipulated into being asymptotic to N (up to the error E), using the fact that fixing $x = 2a+1 \le 2N+1$ fixes the square-free d and the square y^2, namely

$$\sum_{\substack{d \ll N^2 \\ c_d(N) \le 1}} c_d(N) = \sum_{d \ll N^2} \sum_{\substack{1 \le a \le N \\ 1 \le y \le \frac{N}{2}}} \chi_{(2a+1)^2 - dy^2 = 1} + O(E) = \sum_{a=1}^{N} \sum_{d,y} \chi_{(2a+1)^2 - 1 = dy^2} + O(E) = N + O(E).$$

Here $\chi_\cdot$ denotes the characteristic function (taking the value 0 if $\cdot$ is not satisfied, and 1 otherwise).

Now we bound the error sum E. Note that solutions to Pell's equation $x^2 - dy^2 = 1$ grow exponentially, hence we have $c_d(N) \ll \log N$. This means we may assume that $N \gg d \gg N^{1-\delta}$ for some small enough fixed $\delta > 0$, since the contribution of $d \ll N^{1-\delta}$ is bounded by $N^{1-\delta}\log N$. By $x^2 - dy^2 = 1$, we have that $d \gg N^{1-\delta}$ implies $y \ll N^{1/2+\delta}$.

Fixing each $y \ll N^{\delta/2}$ gives $\ll N^{1-\delta}$ choices for x (hence also for d). So we may assume $N^{1/2+\delta} \gg y \gg N^{\delta/2}$, since the contribution of $y \ll N^{\delta/2}$ is bounded by $N^{1-\delta/2}$.

By placing x in residue classes modulo y^2 and splitting the interval $[1, 2N+1]$ into intervals of length y^2, we get that each choice of $N^{\delta/2} \ll y \ll N^{1/2+\delta}$ gives $\ll Ng(y)/y^2$ choices for x (hence also for d) by $y^2 \mid x^2 + 1$, where $g(y) = |\{1 \le x \le y^2 : x^2 + 1 \equiv 0 \pmod{y^2}\}|$. By elementary number theory, g is multiplicative and $g(p^k) \le 2$ for all prime powers p^k. In particular we obtain

$g(n) \leq \tau(n) \ll n^\varepsilon$ for any $\varepsilon > 0$ (this is not hard to prove directly for g, but may be used as a well-known fact for the divisor function τ). Therefore the contribution of such y is

$$\ll \sum_{N^{\delta/2} \ll y \ll N^{1/2+\delta}} \frac{Ng(y)}{y^2} \ll N^{1-\delta/2+\varepsilon},$$

which is acceptable by choosing $\varepsilon > 0$ small enough. We conclude that $S_N = N(1 + o(1))$, as desired.

Remark. There is a secondary infinite family of solutions of "size" $\sqrt{N}$, namely given by $a = 4b(b+1)$. This shows that

$$\limsup_{N \to \infty} \frac{S_N - N}{\sqrt{N}} > 0.$$