IMC 2025

Second Day, July 31, 2025

Problem 6. Let $f\colon (0,\infty) \to \mathbb{R}$ be a continuously differentiable function, and let $b > a > 0$ be real numbers such that $f(a) = f(b) = k$. Prove that there exists a point $\xi \in (a,b)$ such that

$$f(\xi) - \xi f'(\xi) = k.$$

(10 points)

Problem 7. Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all nonempty subsets $M \subseteq \mathbb{Z}_{>0}$ satisfying both of the following properties:

(a) if $x \in M$, then $2x \in M$,

(b) if $x, y \in M$ and $x + y$ is even, then $\dfrac{x+y}{2} \in M$.

(10 points)

Problem 8. For an $n \times n$ real matrix $A \in M_n(\mathbb{R})$, denote by A^{R} its counter-clockwise $90°$ rotation. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{\mathsf{R}} = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 8 \\ 1 & 4 & 7 \end{bmatrix}.$$

Prove that if $A = A^{\mathsf{R}}$ then for any eigenvalue λ of A, we have $\operatorname{Re}\lambda = 0$ or $\operatorname{Im}\lambda = 0$.

(10 points)

Problem 9. Let n be a positive integer. Consider the following random process which produces a sequence of n distinct positive integers $X_1, X_2, \ldots, X_n$.

First, X_1 is chosen randomly with $\mathbb{P}(X_1 = i) = 2^{-i}$ for every positive integer i. For $1 \le j \le n-1$, having chosen $X_1, \ldots, X_j$, arrange the remaining positive integers in increasing order as $n_1 < n_2 < \cdots$, and choose X_{j+1} randomly with $\mathbb{P}(X_{j+1} = n_i) = 2^{-i}$ for every positive integer i.

Let $Y_n = \max\{X_1, \ldots, X_n\}$. Show that

$$\mathbb{E}[Y_n] = \sum_{i=1}^{n} \frac{2^i}{2^i - 1}$$

where $\mathbb{E}[Y_n]$ is the expected value of Y_n.

(10 points)

Problem 10. For any positive integer N, let S_N be the number of pairs of integers $1 \le a, b \le N$ such that the number $(a^2 + a)(b^2 + b)$ is a perfect square. Prove that the limit

$$\lim_{N\to\infty} \frac{S_N}{N}$$

exists and find its value.

(10 points)