IMC 2025

First Day, July 30, 2025

Solutions

Problem 1. Let $P \in \mathbb{R}[x]$ be a polynomial with real coefficients, and suppose $\deg(P) \geq 2$. For every $x \in \mathbb{R}$, let $\ell_x \subset \mathbb{R}^2$ denote the line tangent to the graph of P at the point $(x, P(x))$.

(a) Suppose that the degree of P is odd. Show that $\bigcup_{x\in\mathbb{R}} \ell_x = \mathbb{R}^2$.

(b) Does there exist a polynomial of even degree for which the above equality still holds?

(proposed by Mike Daas, Max Planck Institute for Mathematics, Bonn)

Solution.

(a) Suppose that the degree of P is odd and let $(a, b) \in \mathbb{R}$ be arbitrary. Given $r \in \mathbb{R}$, the equation for ℓ_r is given by

$$\ell_r = \left\{(x, y) \in \mathbb{R}^2 \mid y = P'(r)(x - r) + P(r)\right\}.$$

For this line to pass through the point (a, b) it is therefore necessary and sufficient that

$$b = aP'(r) + P(r) - rP'(r).$$

This is a polynomial equation in r, which always has a real solution as soon as we can show that the degree is odd. Indeed, if $P(r) = cr^n + \ldots$ describes the leading term, then the right hand side of the above equation has leading term $(c - nc)r^n$. Since $c \neq 0$ and we assumed that $n \geq 2$, we must have $c - nc \neq 0$. The right hand side therefore has the same degree as P, completing the proof.

(b) If the degree of P is even, then this can never be true, because the degree of $aP'(r)+P(r)-rP'(r)$ is now even by the same argument and therefore it has a global minimum (or maximum) over the reals, below (or above) which no value of b will yield a real solution for r.

Problem 2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function, and suppose that $\int_{-1}^{1} f(x)\,\mathrm{d}x = 0$ and $f(1) = f(-1) = 1$. Prove that

$$\int_{-1}^{1} \left(f''(x)\right)^2 \,\mathrm{d}x \ge 15,$$

and find all such functions for which equality holds.

(proposed by Alberto Cagnetta, Università degli Studi di Udine, Italy)

Solution. If g is an arbitrary twice continously differentable function on $[-1,1]$, then by applying the Caucy–Schwarz inequality for f'' and g and integrate by parts twice we get

$$\begin{aligned}\sqrt{\int_{-1}^{1}(f'')^2 \cdot \int_{-1}^{1} g^2} &\ge \int_{-1}^{1} f''g = \big(f'(1)g(1) - f'(-1)g(-1)\big) - \int_{-1}^{1} f'g' = \\ &= \big(f'(1)g(1) - f'(-1)g(-1)\big) - \big(f(1)g'(1) - f(-1)g'(-1)\big) + \int_{-1}^{1} fg'' \\ &= \big(f'(1)g(1) - f'(-1)g(-1) - g'(1) + g'(-1)\big) + \int_{-1}^{1} fg''. \qquad (1)\end{aligned}$$

In order to get rid of the terms $f'(1)g(1)$ and $f'(-1)g(-1)$ we will chose g such that $g(1) = g(-1) = 0$. Moreover, if g'' is constant, so g is an at most quadratic polynomial, then $\int_{-1}^{1} fg'' = g'' \int_{-1}^{1} = f0$. Hence, it is reasonable to apply (1) with $g(x) = (1+x)(1-x) = 1 - x^2$.

With this choice,

$$g(1) = g(-1) = 0, \quad g'(1) = -2, \quad g'(-1) = 2, \quad g'' \equiv -2$$

and

$$\int_{-1}^{1} g^2 = \int_{-1}^{1} (1 - 2x^2 + x^4)\,\mathrm{d}x = \frac{16}{15},$$

so we get

$$\sqrt{\int_{-1}^{1}(f'')^2 \cdot \frac{16}{15}} \ge 0 - 0 - g'(1) + g'(-1) + 0 = 4,$$

$$\int_{-1}^{1} (f'')^2 \ge 15.$$

Equality in Cauchy–Schwarz in (1) holds only if there exists a real λ such that $f''(x) = \lambda g(x)$ almost everywhere in $[-1,1]$; by continuity of f'' and g we have $f'' = \lambda g$ everwhere on the interval $[-1,1]$. Hence $f(x)$ must have the form

$$f(x) = \lambda\left(\frac{x^2}{2} - \frac{x^4}{12}\right) + ax + b, \quad \text{with } \lambda, a, b \in \mathbb{R}.$$

From $\int_{-1}^{1} f(x)\mathrm{d}x = 0$, we get $0 = \frac{3\lambda}{10} + 2b$, hence $b = -\frac{3}{20}\lambda$. Moreover, the condition $f(1) = f(-1) = 1$ implies

$$\lambda\left(\frac{1}{2} - \frac{1}{12}\right) + a - \frac{3}{20}\lambda = f(1) = 1 = f(-1) = \lambda\left(\frac{1}{2} - \frac{1}{12}\right) - a - \frac{3}{20}\lambda$$

hence $a = 0$, and $\lambda = \frac{15}{4}$.

In conclusion, the equality holds if and only if

$$f(x) = \frac{15}{4} \cdot \left(\frac{x^2}{2} - \frac{x^4}{12} - \frac{3}{20}\right) = \frac{-5x^4 + 30x^2 - 9}{16} \quad \text{for all } x \in [-1,1].$$

Problem 3. Denote by $\mathcal{S}$ the set of all real symmetric 2025×2025 matrices of rank 1 whose entries take values -1 or $+1$. Let $A, B \in \mathcal{S}$ be matrices chosen independently uniformly at random. Find the probability that A and B commute, i.e. $AB = BA$.

(proposed by Marian Panţiruc, "Gheorghe Asachi" Technical University of Iaşi, Romania)

Solution. Let $n = 2025$. First, we give a charaterisation of matrices in $\mathcal{S}$.

Suppose that $A = (a_{ij})_{i,j=1}^n \in \mathcal{S}$. Since $\operatorname{rk} A = 1$, for every $1 < i, j \le n$, we have

$$\det \begin{pmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{pmatrix} = a_{11}a_{ij} - a_{i1}a_{1j} = a_{11}a_{ij} - a_{i1}a_{j1} = 0.$$

If $a_{11} = 1$ then this means $a_{ij} = a_{i1}a_{j1}$. In this case, let $u = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$; then we have $A = \big(a_{i1}a_{j1}\big) = uu^\top$. Otherwise, if $a_{11} = -1$, we have $a_{ij} = -a_{i1}a_{j1}$. In that case, let $u = \begin{pmatrix} -a_{11} \\ -a_{21} \\ \vdots \\ -a_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ -a_{21} \\ \vdots \\ -a_{n1} \end{pmatrix}$; then $A = -\big(a_{i1}a_{j1}\big) = -uu^\top$.

Hence, all matrices in $\mathcal{S}$ can uniquely be written as $\pm uu^\top$ with a vector $u = \begin{pmatrix} 1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ such that $u_2, \dots, u_n \in \{\pm 1\}$. (Note that $\operatorname{rk}(uu^\top) = 1$ is satisfied.) In particular, we have $|\mathcal{S}| = 2^n$, because the sign and the coordinates $u_2, \dots, u_n$ can be chosen independently.

Now, if $A = \pm uu^T$ and $B = \pm vv^T$ are elements of $\mathcal{S}$, then

$$AB = \pm(uu^\top)(vv^\top) = \pm u(u^\top v)v^\top = \pm u \cdot \langle u, v\rangle \cdot v^\top = \pm\langle u, v\rangle \cdot \big(u_i v_j\big)_{i,j=1}^n$$

and similarly

$$BA = \pm\langle u, v\rangle \cdot \big(v_i u_j\big)_{i,j=1}^n.$$

Since $n = 2025$ is odd, it follows that $\langle u, v\rangle \neq 0$. The first columns of the matrices $uv^\top = \big(u_i v_j\big)_{i,j=1}^n$ and $vu^\top = \big(v_i u_j\big)_{i,j=1}^n$ are u and v, respectively. Hence, $AB = BA$ if and only if $u = v$; in other words, if $A = \pm B$.

For each $A \in \mathcal{S}$, there are precisely two suitable matrices $B \in \mathcal{S}$, so the probability that A, B commute is $\dfrac{2}{|\mathcal{S}|} = \dfrac{1}{2^{n-1}}$.

Problem 4. Let a be an even positive integer. Find all real numbers x such that

$$\left\lfloor \sqrt[a]{b^a + x} \cdot b^{a-1} \right\rfloor = b^a + \lfloor x/a \rfloor \tag{1}$$

holds for every positive integer b.
(Here $\lfloor x \rfloor$ denotes the largest integer that is no greater than x.)

(proposed by Yagub Aliyev, ADA University, Baku, Azerbaijan)

Solution. We will show that if $a = 2$ then we must have $x \in [-1,\ 2) \cup [3,\ 4)$, otherwise $x \in [-1,\ a)$.

Suppose that $\lfloor x/a \rfloor = m$. Then $m \le x/a < m + 1$, and

$$am \le x < a(m + 1). \tag{2}$$

Let $b = 1$. Then

$$\left\lfloor \sqrt[a]{1 + x} \right\rfloor = 1 + \lfloor x/a \rfloor . \tag{3}$$

From (3) it follows that $\left\lfloor \sqrt[a]{1 + x} \right\rfloor = 1 + m$, or $1 + m \le \sqrt[a]{1 + x} < 2 + m$, or

$$(1 + m)^a - 1 \le x < (2 + m)^a - 1, \tag{4}$$

where we have used the obvious fact that $m \ge -1$. Indeed, the number $\sqrt[a]{1 + x}$ and therefore the number $\left\lfloor \sqrt[a]{1 + x} \right\rfloor = 1 + m$ is not negative. By Bernoulli's inequality, the following two inequalities — which compare the inequalities (2) and (4) — hold:

$$am \le (1 + m)^a - 1, \tag{5}$$
$$a(m + 1) \le (1 + (m + 1))^a - 1, \tag{6}$$

where in (5), equality holds if and only if $m = 0$, and in (6), equality holds if and only if $m = -1$. From (2), (4)–(6), it follows that

$$(m + 1)^a - 1 \le x < a(m + 1). \tag{7}$$

From (7), it follows that $(m + 1)^a - 1 < a(m + 1)$. Therefore

$$(m + 1)^a \ \le\ a(m + 1). \tag{8}$$

From (8), it follows that $m + 1 \le a^{\frac{1}{a-1}}$. If $a > 2$ then $m + 1 < a^{\frac{1}{a-1}} < 2$, because $2^{a-1} > a$ for $a > 2$ (one can prove this by mathematical induction) and $1 < a^{\frac{1}{a-1}} < 2$ is not an integer. Therefore, if $a > 2$ then $m = 0$ or $m = -1$. If $a = 2$ then $m = -1$, $m = 0$ or $m = 1$. From (7), it follows that the equality (3) holds true only for the values $-1 \le x < 0$ $(m = -1)$, $0 \le x < a$ $(m = 0)$ if $a > 2$, and $-1 \le x < 0$ $(m = -1)$, $0 \le x < 2$ $(m = 0)$ and $3 \le x < 4$ $(m = 1)$ if $a = 2$.

We will now prove that for these values of x, the equality (1) is true for all positive integers b. From (7), it follows that $b^a + (m + 1)^a - 1 \le b^a + x < b^a + a(m + 1)$ and

$$1 + \frac{(m + 1)^a - 1}{b^a} \le 1 + \frac{x}{b^a} < 1 + \frac{a(m + 1)}{b^a}. \tag{9}$$

By Bernoulli's inequality

$$1 + \frac{a(m + 1)}{b^a} \le \left(1 + \frac{m + 1}{b^a}\right)^a, \tag{10}$$

where equality occurs if and only if $m = -1$. It is easy to check that for $m = 1$ (if $a = 2$), $m = 0$ and $m = -1$, the following inequality holds true:

$$\left(1 + \frac{m}{b^a}\right)^a \le 1 + \frac{(m + 1)^a - 1}{b^a}. \tag{11}$$

From (9)–(11), it follows that

$$\left(1+\frac{m}{b^a}\right)^a \leq 1+\frac{x}{b^a} < \left(1+\frac{m+1}{b^a}\right)^a. \tag{12}$$

From (12), it follows that

$$b^a + m \leq \sqrt[a]{b^a + x} \cdot b^{a-1} < b^a + m + 1,$$
$$b^a + \lfloor x/a \rfloor \leq \sqrt[a]{b^a + x} \cdot b^{a-1} < b^a + \lfloor x/a \rfloor + 1.$$

Consequently, equality (1) holds.

Problem 5. For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. Denote by S_n the set of all bijections from $[n]$ to $[n]$, and let T_n be the set of all maps from $[n]$ to $[n]$. Define the *order* $\operatorname{ord}(\tau)$ of a map $\tau \in T_n$ as the number of distinct maps in the set $\{\tau, \tau \circ \tau, \tau \circ \tau \circ \tau, \ldots\}$ where $\circ$ denotes composition. Finally, let

$$f(n) = \max_{\tau \in S_n} \operatorname{ord}(\tau) \quad \text{and} \quad g(n) = \max_{\tau \in T_n} \operatorname{ord}(\tau).$$

Prove that $g(n) < f(n) + n^{0.501}$ for sufficiently large n.

(proposed by Fedor Petrov, St Petersburg State University)

Solution. For every $\tau \in T_n$ we need to prove that $\operatorname{ord}(\tau) \leqslant f(n) + n^{0.501}$ (if n is large enough). Denote by $C(\tau)$ the set of elements $x \in [n]$ for which $\tau^k(x) = \underbrace{\tau(\ldots\tau}_{k}(x)\ldots) = x$ for some $k > 0$. It is immediate that $C(\tau)$ is a τ-invariant set; let $\tau_c = \tau|_{C(\tau)}$, that is a permutation on $C(\tau)$.

Let N be the order of this permutation τ_c, clearly $N \leqslant g(n)$. Consider an arbitrary element $x \in [n] \setminus C(\tau)$. The sequence $x, \tau(x), \tau^2(x), \ldots$ is eventually periodic, but not from the beginning, because $x \notin C(\tau)$.

Let $h(x) > 0$ be the minimal number for which $\tau^{h(x)}(x)$ is in the period; equivalently, this is the minimal number for which $\tau^{h(x)}(x) \in C(\tau)$. Let $R = \max\limits_{x \in [n] \setminus C(\tau)} h(x)$. Note that $\tau^R(x) = \tau^{R+N}(x)$ for all $x \in [n]$, since $\tau^R(x) \in C(\tau)$ for all $x \in [n]$. Therefore, $\operatorname{ord}(\tau) \leqslant N + R$. Thus, if $R < n^{0.501}$, we are done.

Now assume that $R \geqslant n^{0.501}$, that is, $h(x) \geqslant n^{0.501}$ for some $x \notin C(\tau)$. It yields $|C(\tau)| \leqslant n - n^{0.501}$. Consider the cycle lengths of the permutation τ_c. We claim that there exists a prime number $p < n^{0.501}$ which does not divide any of these lengths. Indeed, otherwise the sum of cycle lengths of τ_c is not less than the sum of all prime numbers not exceeding $n^{0.501}$ (because each positive integer is not less than the sum of its prime divisors, that in turn follows from $ab \geqslant a + b$ for $a, b \geqslant 2$). But for large n, the number of prime numbers less than $n^{0.501}$ is at least $n^{0.5009}$ by some weak version of Prime Number Theorem (that also has many short proofs). Therefore, their sum exceeds n, contradiction.

Now we may consider the permutation $\tau_0 \in S_n$ which acts as τ_c on $C(\tau)$ and has a cycle of length p on $[n] \setminus C(\tau)$. The order of τ_0 is not less than $p \cdot N$, therefore $N \leqslant f(n)/p \leqslant f(n)/2$, and by the above argument we get $\operatorname{ord}(\tau) \leqslant N + R \leqslant f(n)/2 + n < f(n)$ for large n.