IMC 2025

First Day, July 30, 2025

Problem 1. Let $P \in \mathbb{R}[x]$ be a polynomial with real coefficients, and suppose $\deg(P) \geq 2$. For every $x \in \mathbb{R}$, let $\ell_x \subset \mathbb{R}^2$ denote the line tangent to the graph of P at the point $(x, P(x))$.

(a) Suppose that the degree of P is odd. Show that $\bigcup_{x \in \mathbb{R}} \ell_x = \mathbb{R}^2$.

(b) Does there exist a polynomial of even degree for which the above equality still holds?

(10 points)

Problem 2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function, and suppose that $\int_{-1}^{1} f(x)\,\mathrm{d}x = 0$ and $f(1) = f(-1) = 1$. Prove that

$$\int_{-1}^{1} \left(f''(x)\right)^2 \,\mathrm{d}x \geq 15,$$

and find all such functions for which equality holds.

(10 points)

Problem 3. Denote by $\mathcal{S}$ the set of all real symmetric 2025×2025 matrices of rank 1 whose entries take values -1 or $+1$. Let $A, B \in \mathcal{S}$ be matrices chosen independently uniformly at random. Find the probability that A and B commute, i.e. $AB = BA$.

(10 points)

Problem 4. Let a be an even positive integer. Find all real numbers x such that

$$\left\lfloor \sqrt[a]{b^a + x} \cdot b^{a-1} \right\rfloor = b^a + \lfloor x/a \rfloor \tag{1}$$

holds for every positive integer b.
(Here $\lfloor x \rfloor$ denotes the largest integer that is no greater than x.)

(10 points)

Problem 5. For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. Denote by S_n the set of all bijections from $[n]$ to $[n]$, and let T_n be the set of all maps from $[n]$ to $[n]$. Define the *order* $\operatorname{ord}(\tau)$ of a map $\tau \in T_n$ as the number of distinct maps in the set $\{\tau, \tau \circ \tau, \tau \circ \tau \circ \tau, \ldots\}$ where $\circ$ denotes composition. Finally, let

$$f(n) = \max_{\tau \in S_n} \operatorname{ord}(\tau) \quad \text{and} \quad g(n) = \max_{\tau \in T_n} \operatorname{ord}(\tau).$$

Prove that $g(n) < f(n) + n^{0.501}$ for sufficiently large n.

(10 points)