IMC 2020 Online

Day 2, July 27, 2020

Problem 5. Find all twice continuously differentiable functions $f : \mathbb{R} \to (0, +\infty)$ satisfying

$$f''(x)f(x) \geq 2(f'(x))^2$$

for all $x \in \mathbb{R}$.

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Solution. We shall show that only positive constant functions satisfy the condition.

Let $g(x) = \dfrac{1}{f(x)}$. Notice that

$$g'' = \left(\frac{1}{f}\right)'' = \left(\frac{-f'}{f^2}\right)' = \left(\frac{2(f')^2 - f''f}{f^3}\right)' \leq 0,$$

so the positive function $g(x)$ is concave. We show that g must be constant.

Take two arbitrary real numbers $a < b$. By the concavity of g, for all $u < a$ and $v > b$ we have

$$\frac{g(a) - g(u)}{a - u} \geq \frac{g(b) - g(a)}{b - a} \geq \frac{g(v) - g(b)}{v - b}.$$

Combining this with $g(u), g(v) > 0$ we get

$$\frac{g(a)}{a - u} > \frac{g(b) - g(a)}{b - a} > \frac{-g(b)}{v - b}$$

Now by taking limits $u \to -\infty$ and $v \to \infty$ we obtain

$$0 \geq \frac{g(b) - g(a)}{b - a} \geq 0,$$

so $g(a) = g(b)$. This holds for any pair (a, b), so $g(x)$ is constant and $f(x) = 1/g(x)$ also is constant.

If f is constant then $f' = f'' = 0$, so the condition is satisfied.

Remark. Instead of the function $1/f(x)$, the same idea works with $\arctan f(x)$:

$$(\arctan f(x))'' = \frac{f''(1 + f^2) - 2(f')^2}{(1 + f^2)^2} = \frac{f''(1 + f^2) - 2(f')^2(1 + f^2)}{(1 + f^2)^2} = \frac{f'' - 2(f')^2}{1 + f^2} \geq 0.$$

As can be seen, $\arctan f(x)$ is a bounded convex function, therefore it must be constant.

Problem 6. Find all prime numbers p for which there exists a unique $a \in \{1, 2, \ldots, p\}$ such that $a^3 - 3a + 1$ is divisible by p.

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Solution 1. We show that $p = 3$ the only prime that satsfies the condition.

Let $f(x) = x^3 - 3x + 1$. As preparation, let's compute the roots of $f(x)$. By Cardano's formula, it can be seen that the roots are

$$2\mathrm{Re}\sqrt[3]{\frac{-1}{2} + \sqrt{\left(\frac{-1}{2}\right)^2 - \left(\frac{-3}{3}\right)^3}} = 2\mathrm{Re}\sqrt[3]{\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}} = \left\{2\cos\frac{2\pi}{9}, 2\cos\frac{4\pi}{9}, 2\cos\frac{8\pi}{9}\right\}$$

where all three values of the complex cubic root were taken.

Notice that, by the trigonometric identity $2\cos 2t = (2\cos t)^2 - 2$, the map $\varphi(x) = x^2 - 2$ cyclically permutes the three roots. We will use this map to find another root of f, when it is considered over $\mathbb{F}_p$.

Suppose that $f(a) = 0$ for some $a \in \mathbb{F}_p$ and consider

$$g(x) = \frac{f(x)}{x-a} = \frac{f(x) - f(a)}{x-a} = x^2 + ax + (a^2 - 3).$$

We claim that $b = a^2 - 2$ is a root of $g(x)$. Indeed,

$$g(b) = (a^2-2)^2 + a(a^2-2) + (a^2-3) = (a+1)\cdot f(a) = 0.$$

By Vieta's formulas, the other root of $g(x)$ is $c = -a - b = -a^2 - a + 2$.

If f has a single root then the three roots must coincide, so

$$a = a^2 - 2 = -a^2 - a + 2.$$

Here the quadratic equation $a = a^2 - 2$, or equivalently $(a+1)(a-2) = 0$, has two solutions, $a = -1$ and $a = 2$. By $f(-1) = f(2) = 3$, in both cases we have $0 = f(a) = 3$, so the only choice is $p = 3$.

Finally, for $p = 3$ we have $f(1) = -1$, $f(2) = 3$ and $f(3) = 19$, from these values only $f(2)$ is divisible by 3, so $p = 3$ satisfies the condition.

Solution 2 (outline) Define $f(x)$ and $g(x)$ like in Solution 1. The discriminant of $g(x)$ is

$$\Delta_g = a^2 - 4(a^2 - 3) = 12 - 3a^2.$$

We show that Δ_g has a square root in $\mathbb{F}_p$.

Take two integers k, m (to be determinated later) and consider

$$\Delta_g = \Delta_g + (ka + m)f(a) = ka^4 + ma^3 - (3k+1)a^2 + (k - 3m)a + (m + 12).$$

Our goal is to choose k, m in such a way that the last expression is a complete square. Either by direct calculations or guessing, we can find that $k = m = 4$ works:

$$\Delta_g = \Delta_g + (4a+4)f(a) = 4a^4 + 4a^3 - 15a^2 - 8a + 16 = (2a^2 + a - 4)^2.$$

If $p \neq 2$ then we can conclude that $f(x)$ has either no or three roots, therefore p is suitable if and only is $f(x)$ is a complete cube: $x^3 - 3x + 1 = (x - a)^3$. From Vieta's formulas $a^3 = 1$, so $a \neq 0$ and $3a = 0$, which is possible if $p = 3$.

For $p = 3$ we have $f(x) = (x+1)^3$, so $p = 3$ is suitable.

The case $p = 2$ must be checked separately because the quadratic formula contains a division by 2. $f(1) = -1$ and $f(2) = 3$, so $p = 2$ is not suitable.

Solution 3 (outline) Assume $p > 3$; the cases $p = 2$ and $p = 3$ will be checked separately.

Let $f(x) = x^3 - 3x + 1$ and suppose that $a \in \mathbb{F}_p$ is a root of $f(x)$, and let $b, c \in \mathbb{F}_{p^2}$ be the other two roots. The discriminant Δ_f of $f(x)$ can be expressed by the elementary symmetric polynomials of a, b, c; it can be calculated that

$$\Delta_f = (b-c)^2(a-b)^2(a-c)^2 = 81 = 9^2,$$

so

$$(b-c)(a-b)(a-c) = \pm 9 \in \mathbb{F}_p.$$

Notice that $\Delta_f \neq 0$, so the three roots are distinct.

Either $b, c \in \mathbb{F}_p$ or b, c are conjugate elements in $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$, we have $(a-b)(a-c) \in \mathbb{F}_p$, so $b - c = \frac{(b-c)(a-b)(a-c)}{(a-b)(a-c)} \in \mathbb{F}_p$. From Vieta's formulas we have $b + c \in \mathbb{F}_p$ as well; since $p \neq 2$, it follows that $b, c \in \mathbb{F}_p$. Now $f(x)$ has three distinct roots in $\mathbb{F}_p$, so p cannot be suitable.

$p = 2$ does not satisfies the condition because both $f(1) = -1$ and $f(2) = 3$ are odd. $p = 3$ is suitable, because $f(2) = 3$ is divisible by 3 while $f(1) = -1$ and $f(3) = 19$ are not.

Problem 7. Let G be a group and $n \geq 2$ be an integer. Let H_1 and H_2 be two subgroups of G that satisfy

$$[G : H_1] = [G : H_2] = n \quad \text{and} \quad [G : (H_1 \cap H_2)] = n(n-1).$$

Prove that H_1 and H_2 are conjugate in G.

(Here $[G : H]$ denotes the *index* of the subgroup H, i.e. the number of distinct left cosets xH of H in G. The subgroups H_1 and H_2 are *conjugate* if there exists an element $g \in G$ such that $g^{-1} H_1 g = H_2$.)

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Solution 1. Denote $K = H_1 \cap H_2$. Since

$$n(n-1) = [G : K] = [G : H_1][H_1 : K] = n[H_1 : K],$$

we obtain that $[H_1 : K] = n - 1$. Thus, the subgroup H_1 is partitioned into $n - 1$ left cosets of K, say $H_1 = \bigsqcup_{i=1}^{n-1} h_i K$. Therefore, the set $H_1 H_2 = \{ab \colon a \in H_1,\ b \in H_2\}$ is partitioned as

$$H_1 H_2 = \left(\bigsqcup_{i=1}^{n-1} h_i K \right) H_2 = \bigsqcup_{i=1}^{n-1} h_i K H_2 = \bigsqcup_{i=1}^{n-1} h_i H_2.$$

The last equality holds because $K \subseteq H_2$, so $K H_2 = H_2$. The last expression is a disjoint union since

$$h_i H_2 \cap h_j H_2 \neq \varnothing \iff h_i^{-1} h_j \in H_2 \iff h_i^{-1} h_j \in K \iff h_i = h_j.$$

Thus, $H_1 H_2$ is a disjoint union of $n-1$ left cosets with respect to H_2; hence $L = G \setminus (H_1 H_2)$ is the remaining such left coset. Similarly, L is a right coset with respect to H_1. Therefore, for each $g \in L$ we have $L = g H_2 = H_1 g$, which yields $H_2 = g^{-1} H_1 g$.

Solution 2. Put $G/H_1 = X$ and $G/H_2 = Y$, those are n-element sets acted on by G from the left. Let G act on $X \times Y$ from the left coordinate-wise, consider this product as a table, with rows being copies of X and columns being copies of Y.

The stabilizer of a point (x, y) in $X \times Y$ is $H_1 \cap H_2$. By the orbit-stabilizer theorem, we obtain that the orbit Z of (x, y) has size $[G : H_1 \cap H_2] = n(n-1)$.

If Z contains a whole column then there is a subgroup G_1 of G that stabilizes x and acts transitively on Y. If we conjugate G_1 to a group G_1', then its action remains transitive on Y, so by conjugation we obtain columns of the table. Since G acts transitively on X, we cover all the columns. It follows that $Z = X \times Y$, so

$$n(n-1) = |Z| = |X \times Y| = n^2,$$

which is a contradiction.

Hence every column of $X \times Y$ has an element not from Z. The same holds for the rows of $X \times Y$. There are n elements not from Z in total and they induce a bijection between X and Y which allows us to identify $X = Y$.

After this identification, every element (x, x) from the diagonal of $X \times X$ (i.e. from $(X \times X) \setminus Z$) is moved to a diagonal element by any $g \in G$, because $gx = gx$. In this formula the action of g in the left hand side and the action of g in the right hand side are the actions of g on X and Y respectively.

Therefore our bijection between X and Y is an isomorphism of sets with a left action of G. Since H_1 and H_2 are stabilizers of the points in the same transitive action of G, we conclude that they are conjugate.

Remark. The situation in the problem is possible for every $n \geq 2$: let $G = S_n$ and let H_1 an H_2 be the stabilizer subgroups of two elements.

Problem 8. Compute
$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} (-1)^k \binom{n}{k} \log k.$$
(Here log denotes the natural logarithm.)

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Solution 1. Answer: 1.

The idea is that if $f(k) = \int g^k$, then
$$\sum (-1)^k \binom{n}{k} f(k) = \int (1-g)^n.$$
To relate this to logarithm, we may use the Frullani integrals
$$\int_0^\infty \frac{e^{-x} - e^{-kx}}{x} dx = \lim_{c\to+0} \int_c^\infty \frac{e^{-x}}{x} dx - \int_c^\infty \frac{e^{-kx}}{x} dx = \lim_{c\to+0} \int_c^\infty \frac{e^{-x}}{x} dx - \int_{kc}^\infty \frac{e^{-x}}{x} dx =$$
$$\lim_{c\to+0} \int_c^{kc} \frac{e^{-x}}{x} dx = \log k + \lim_{c\to+0} \int_c^{kc} \frac{e^{-x}-1}{x} dx = \log k.$$
This gives the integral representation of our sum:
$$A := \sum_{k=1}^{n} (-1)^k \binom{n}{k} \log k = \int_0^\infty \frac{-e^{-x} + 1 - (1-e^{-x})^n}{x} dx.$$
Now the problem is reduced to a rather standard integral asymptotics.

We have $(1-e^{-x})^n \geqslant 1 - ne^{-x}$ by Bernoulli inequality, thus $0 \leqslant -e^{-x} + 1 - (1-e^{-x})^n \leqslant ne^{-x}$, and we get
$$0 \leqslant \int_M^\infty \frac{-e^{-x} + 1 - (1-e^{-x})^n}{x} dx \leqslant n \int_M^\infty \frac{e^{-x}}{x} dx \leqslant nM^{-1} \int_M^\infty e^{-x} dx = nM^{-1}e^{-M}.$$
So choosing M such that $Me^M = n$ (such M exists and goes to ∞ with n) we get
$$A = O(1) + \int_0^M \frac{-e^{-x} + 1 - (1-e^{-x})^n}{x} dx.$$
Note that for $0 \leqslant x \leqslant M$ we have $e^{-x} \geqslant e^{-M} = M/n$, and $(1-e^{-x})^{n-1} \leqslant e^{-e^{-x}(n-1)} \leqslant e^{-M(n-1)/n}$ tends to 0 uniformly in x. Therefore
$$\int_0^M \frac{(1-e^{-x})(1-(1-e^{-x})^{n-1})}{x} dx = (1+o(1)) \int_0^M \frac{1-e^{-x}}{x} dx.$$
Finally
$$\int_0^M \frac{1-e^{-x}}{x} dx = \int_0^1 \frac{1-e^{-x}}{x} dx + \int_1^M \frac{-e^{-x}}{x} dx + \int_1^M \frac{dx}{x} =$$
$$\log M + O(1) = \log(M + \log M) + O(1) = \log\log n + O(1),$$
and we get $A = (1+o(1)) \log\log n$.

Solution 2. We start with a known identity (a finite difference of $1/x$).

Expand the rational function

$$f(x) = \frac{m!}{x(x+1)\dots(x+m)}$$

as the linear combination of simple fractions $f(x) = \sum_{j=0}^{m} c_j/(x+j)$. To find c_j we use

$$c_j = ((x+j)f(x))\,|_{x=-j} = (-1)^j \binom{m}{j}.$$

So we get

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{1}{x+k} = \frac{m!}{x(x+1)\dots(x+m)}. \tag{1}$$

Another known identity we use is

$$\sum_{k=j+1}^{n} (-1)^k \binom{n}{j} = \sum_{k=j+1}^{n} (-1)^k \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right) = (-1)^{j+1} \binom{n-1}{j}. \tag{2}$$

Finally we write $\log k = \int_1^k \frac{dx}{x} = \sum_{j=1}^{k-1} I_j$, where $I_j = \int_0^1 \frac{dx}{x+j}$.

Now we have

$$S := \sum_{k=1}^{n} (-1)^k \binom{n}{k} \log k = \sum_{k=1}^{n} (-1)^k \binom{n}{k} \sum_{j=1}^{k-1} I_j = \sum_{j=1}^{n-1} I_j \sum_{k=j+1}^{n} (-1)^k \binom{n}{k} \underset{(2)}{=} \sum_{j=1}^{n-1} I_j (-1)^{j+1} \binom{n-1}{j} =$$

$$\int_0^1 \sum_{j=1}^{n-1} (-1)^{j+1} \binom{n-1}{j} \frac{dx}{x+j} = \int_0^1 \left(\frac{1}{x} - \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \frac{dx}{x+j} \right) dx \underset{(1)}{=}$$

$$\int_0^1 \left(\frac{1}{x} - \frac{(n-1)!}{x(x+1)\dots(x+(n-1))} \right) dx = \int_0^1 \frac{dx}{x} \left(1 - \frac{1}{(1+x)(1+x/2)\dots(1+x/(n-1))} \right).$$

So S is again expressed as an integral, for which it is not hard to get an asymptotics.

Since $e^t \geqslant 1+t$ for all real t (by convexity or any other reason), we have $e^{y^2-y} \geqslant 1+y^2-y = \frac{1+y^3}{1+y} \geqslant \frac{1}{1+y}$ and $\frac{1}{1+y} \geqslant \frac{1}{e^y} = e^{-y}$ for $y>0$. Therefore

$$e^{y^2-y} \geqslant \frac{1}{1+y} \geqslant e^{-y},\ y>0.$$

Using this double inequality we get

$$e^{x^2\left(1+\frac{1}{2^2}+\dots+\frac{1}{(n-1)^2}\right) - x\left(1+\frac{1}{2}+\dots+\frac{1}{n-1}\right)} \geqslant \frac{1}{(1+x)(1+x/2)\dots(1+x/(n-1))} \geqslant e^{-x\left(1+\frac{1}{2}+\dots+\frac{1}{n-1}\right)}.$$

Since $x^2(1+1/2^2+\dots) \leqslant 2x^2 \leqslant 2x$, we conclude that

$$\frac{1}{(1+x)(1+x/2)\dots(1+x/(n-1))} = e^{-C_n x}, \text{where } -2 + \sum_{j=1}^{n-1} \frac{1}{j} \leqslant C_n \leqslant \sum_{j=1}^{n-1} \frac{1}{j},$$

i.e., $C_n = \log n + O(1)$. Thus

$$S = \int_0^1 \frac{dx}{x}(1-e^{-C_n x}) = \int_0^{C_n} \frac{dt}{t}(1-e^{-t}) = \int_1^{C_n} \frac{dt}{t} + \int_0^1 (1-e^{-t})\frac{dt}{t} + \int_1^{C_n} e^{-t}\frac{dt}{t}$$
$$= \log C_n + O(1) = \log\log n + O(1).$$