IMC 2019, Blagoevgrad, Bulgaria

Day 2, July 31, 2019

Problem 6. Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous functions such that g is differentiable. Assume that $\big(f(0) - g'(0)\big)\big(g'(1) - f(1)\big) > 0$. Show that there exists a point $c \in (0, 1)$ such that $f(c) = g'(c)$.

(10 points)

Problem 7. Let $C = \{4, 6, 8, 9, 10, \ldots\}$ be the set of composite positive integers. For each $n \in C$ let a_n be the smallest positive integer k such that $k!$ is divisible by n. Determine whether the following series converges:

$$\sum_{n \in C} \left(\frac{a_n}{n}\right)^n.$$

(10 points)

Problem 8. Let $x_1, \ldots, x_n$ be real numbers. For any set $I \subset \{1, 2, \ldots, n\}$ let $s(I) = \sum\limits_{i \in I} x_i$. Assume that the function $I \mapsto s(I)$ takes on at least 1.8^n values where I runs over all 2^n subsets of $\{1, 2, \ldots, n\}$. Prove that the number of sets $I \subset \{1, 2, \ldots, n\}$ for which $s(I) = 2019$ does not exceed 1.7^n.

(10 points)

Problem 9. Determine all positive integers n for which there exist $n \times n$ real invertible matrices A and B that satisfy $AB - BA = B^2 A$.

(10 points)

Problem 10. 2019 points are chosen at random, independently, and distributed uniformly in the unit disc $\{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \le 1\}$. Let C be the convex hull of the chosen points. Which probability is larger: that C is a polygon with three vertices, or a polygon with four vertices?

(10 points)