IMC 2019, Blagoevgrad, Bulgaria

Day 1, July 30, 2019

Problem 1. Evaluate the product
$$\prod_{n=3}^{\infty} \frac{(n^3+3n)^2}{n^6-64}.$$

Proposed by Orif Ibrogimov, ETH Zurich and National University of Uzbekistan and Karen Keryan, Yerevan State University and American University of Armenia, Yerevan

Hint: Telescoping product.

Solution. Let
$$a_n = \frac{(n^3+3n)^2}{n^6-64}.$$

Notice that
$$\begin{aligned} a_n &= \frac{(n^3+3n)^2}{(n^3-8)(n^3+8)} = \frac{n^2(n^2+3)^2}{(n-2)(n^2+2n+4)\cdot(n+2)(n^2-2n+4)} \\ &= \frac{n}{n-2}\cdot\frac{n}{n+2}\cdot\frac{n^2+3}{(n-1)^2+3}\cdot\frac{n^2+3}{(n+1)^2+3}. \end{aligned}$$

Hence, for $N \geq 3$ we have
$$\begin{aligned} \prod_{n=3}^{N} a_n &= \left(\prod_{n=3}^{N}\frac{n}{n-2}\right)\left(\prod_{n=3}^{N}\frac{n}{n+2}\right)\left(\prod_{n=3}^{N}\frac{n^2+3}{(n-1)^2+3}\right)\left(\prod_{n=3}^{N}\frac{n^2+3}{(n+1)^2+3}\right) \\ &= \frac{N(N-1)}{1\cdot 2}\cdot\frac{3\cdot 4}{(N+1)(N+2)}\cdot\frac{N^2+3}{2^2+3}\cdot\frac{3^2+3}{(N+1)^2+3} \\ &= \frac{72}{7}\cdot\frac{N(N-1)(N^2+3)}{(N+1)(N+2)\big((N+1)^2+3\big)} \\ &= \frac{72}{7}\cdot\frac{(1-\frac{1}{N})(1+\frac{3}{N^2})}{(1+\frac{1}{N})(1+\frac{2}{N})\big((1+\frac{1}{N})^2+\frac{3}{N^2}\big)}, \end{aligned}$$

so
$$\prod_{n=3}^{\infty} a_n = \lim_{N\to\infty}\prod_{n=3}^{N} a_n = \lim_{N\to\infty}\left(\frac{72}{7}\cdot\frac{(1-\frac{1}{N})(1+\frac{3}{N^2})}{(1+\frac{1}{N})(1+\frac{2}{N})\big((1+\frac{1}{N})^2+\frac{3}{N^2}\big)}\right) = \frac{72}{7}.$$

Problem 2. A four-digit number $YEAR$ is called *very good* if the system
$$\begin{aligned} Yx+Ey+Az+Rw &= Y \\ Rx+Yy+Ez+Aw &= E \\ Ax+Ry+Yz+Ew &= A \\ Ex+Ay+Rz+Yw &= R \end{aligned}$$

of linear equations in the variables x, y, z and w has at least two solutions. Find all very good YEARs in the 21st century.

(The 21st century starts in 2001 and ends in 2100.)

Proposed by Tomáš Bárta, Charles University, Prague

Hint: If the solution of the system is not unique then $\det \begin{pmatrix} Y & E & A & R \\ R & Y & E & A \\ A & R & Y & E \\ E & A & R & Y \end{pmatrix} = 0.$

Solution. Let us apply row transformations to the augmented matrix of the system to find its rank. First we add the second, third and fourth row to the first one and divide by $Y+E+A+R$ to get

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ R & Y & E & A & E \\ A & R & Y & E & A \\ E & A & R & Y & R \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & Y-R & E-R & A-R & E-R \\ 0 & R-A & Y-A & E-A & 0 \\ 0 & A-E & R-E & Y-E & R-E \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & Y-R & E-R & A-R & E-R \\ 0 & R-A & Y-A & E-A & 0 \\ 0 & A-E+Y-R & 0 & Y-E+A-R & 0 \end{pmatrix}$$

Let us first omit the last column and look at the remaining 4×4 matrix. If $E \neq R$, the first and second rows are linearly independent, so the rank of the matrix is at least 2 and rank of the augmented 4×5 matrix cannot be bigger than rank of the 4×4 matrix due to the zeros in the last column.

IF $E = R$, then we have three zeros in the last column, so rank of the 4×5 matrix cannot be bigger than rank of the 4×4 matrix. So, the original system has always at least one solution.

It follows that the system has more than one solution if and only if the 4×4 matrix (with the last column omitted) is singular. Let us first assume that $E \neq R$. We apply one more transform to get

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & Y-R & E-R & A-R \\ 0 & (R-A)(E-R)-(Y-R)(Y-A) & 0 & (E-A)(E-R)-(A-R)(Y-A) \\ 0 & A-E+Y-R & 0 & Y-E+A-R \end{pmatrix}$$

Obviously, this matrix is singular if and only if $A - E + Y - R = 0$ or the two expressions in the third row are equal, i.e.

$$RE - R^2 - AE + AR - Y^2 + RY + AY - AR = E^2 - AE - ER + AR - AY + RY + A^2 - AR$$

$$0 = (E-R)^2 + (A-Y)^2,$$

but this is impossible if $E \neq R$. If $E = R$, we have

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & Y-R & 0 & A-R \\ 0 & R-A & Y-A & R-A \\ 0 & A+Y-2R & 0 & Y+A-2R \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & Y-R & 0 & A-R \\ 0 & R-A & Y-A & R-A \\ 0 & A-R & 0 & Y-R \end{pmatrix}.$$

If $A = Y$, this matrix is singular. If $A \neq Y$, the matrix is regular if and only if $(Y-R)^2 \neq (A-R)^2$ and since $Y \neq A$, it means that $Y - R \neq -(A-R)$, i.e. $Y + A \neq 2R$. We conclude that YEAR is very good if and only if

1. $E \neq R$ and $A + Y = E + R$, or
2. $E = R$ and $Y = A$, or
3. $E = R$, $A \neq Y$ and $Y + A = 2R$.

We can see that if $Y = 2$, $E = 0$, then the very good years satisfying 1 are $A+2 = R \neq 0$, i.e. 2002, 2013, 2024, 2035, 2046, 2057, 2068, 2079, condition 2 is satisfied for 2020 and condition 3 never satisfied.

Problem 3. Let $f : (-1, 1) \to \mathbb{R}$ be a twice differentiable function such that

$$2f'(x) + xf''(x) \geq 1 \quad \text{for } x \in (-1, 1).$$

Prove that

$$\int_{-1}^{1} xf(x)\,\mathrm{d}x \geq \frac{1}{3}.$$

Proposed by Orif Ibrogimov, ETH Zurich and National University of Uzbekistan and Karim Rakhimov, Scuola Normale Superiore and National University of Uzbekistan

Hint: $2f'(x) + xf''(x)$ is the second derivative of a certain function.

Solution. Let

$$g(x) = xf(x) - \frac{x^2}{2}.$$

Notice that

$$g''(x) = 2f'(x) + xf''(x) - 1 \geq 0,$$

so g is convex. Estimate g by its tangent at 0: let $g'(0) = a$, then

$$g(x) = g(0) + g'(0)x = ax$$

and therefore

$$\int_{-1}^{1} xf(x)\,\mathrm{d}x = \int_{-1}^{1} \left(g(x) + \frac{x^2}{2}\right)\mathrm{d}x \geq \int_{-1}^{1} \left(ax + \frac{x^2}{2}\right)\mathrm{d}x = \frac{1}{3}.$$

Problem 4. Define the sequence $a_0, a_1, \ldots$ of numbers by the following recurrence:

$$a_0 = 1, \quad a_1 = 2, \quad (n+3)a_{n+2} = (6n+9)a_{n+1} - na_n \quad \text{for } n \geq 0.$$

Prove that all terms of this sequence are integers.

Proposed by Khakimboy Egamberganov, ICTP, Italy

Hint: Determine the generating function $\sum a_n x^n$.

Solution. Take the generating function of this sequence

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

It is easy to see that the sequence is increasing and

$$\frac{a_{n+1}}{a_n} = \frac{(6n+3)a_n - (n-1)a_{n-1}}{(n+2)a_n} < \frac{6n+3}{n+2} \quad \Rightarrow \quad \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} \leq 6.$$

So the generating function converges in some neighbourhood of 0. Then, we have

$$f(x) = 1+2x+\sum_{n=2}^{\infty} a_n x^n = 1+2x+\sum_{n=0}^{\infty} a_{n+2}x^{n+2} = 1+2x+\sum_{n=0}^{\infty} \frac{6n+9}{n+3}a_{n+1}x^{n+2} - \sum_{n=0}^{\infty} \frac{n}{n+3}a_n x^{n+2}.$$

Let $f_1(x) = \sum_{n=0}^{\infty} \frac{6n+9}{n+3} a_{n+1} x^{n+2}$ and $f_2(x) = \sum_{n=0}^{\infty} \frac{n}{n+3} a_n x^{n+2}$. Then

$$(xf_1(x))' = \sum_{n=0}^{\infty}(6n+9)a_{n+1}x^{n+2} = 6x^2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 3x\sum_{n=0}^{\infty}a_{n+1}x^{n+1} = 6x^2f'(x)+3x(f(x)-1)$$

and

$$(xf_2(x))' = \sum_{n=0}^{\infty} na_nx^{n+2} = x^2\sum_{n=0}^{\infty}(n+1)a_nx^n - x^2\sum_{n=0}^{\infty}a_nx^n = x^2(xf(x))' - x^2f(x) = x^3f'(x).$$

Using this relations, we arrive at the following differential equation for f:

$$(xf(x))' = 1 + 4x + (xf_1(x))' - (xf_2(x))' = 1 + x + (6x^2 - x^3)f'(x) + 3xf(x)$$

or, equivalently,

$$(x^3 - 6x^2 + x)f'(x) + (1 - 3x)f(x) - 1 - x = 0.$$

So, we need solve this differential equation in some sufficiently smaller neighbourhood of 0. We know that $f(0) = 1$ and we need a neighbourhood of 0 such that $x^2 - 6x + 1 > 0$. Then

$$f'(x) + \frac{1-3x}{x(x^2-6x+1)}f(x) = \frac{1+x}{x(x^2-6x+1)}$$

for $x \neq 0$. So the integral multiplier is $\mu(x) = \frac{x}{\sqrt{x^2-6x+1}}$ and

$$(f(x)\mu(x))' = \frac{x+x^2}{(x^2-6x+1)^{\frac{3}{2}}},$$

so

$$f(x) = \left(\frac{1-x}{2\sqrt{x^2-6x+1}} - \frac{1}{2}\right)\frac{\sqrt{x^2-6x+1}}{x} = \frac{1-x-\sqrt{x^2-6x+1}}{2x}.$$

We found the generating function of (a_n) in some neighbourhood of 0, which $x^2 - 6x + 1 > 0$. So our series uniformly converges to $f(x) = \frac{1-x-\sqrt{x^2-6x+1}}{2x}$ in $|x| < 3 - 2\sqrt{2}$.

Instead of computing the coefficients of the Taylor series of $f(x)$ directly, we will find another recurrence relation for (a_n). It is easy to see that $f(x)$ satisfies the quadratic equation $xt^2 - (1-x)t + 1 = 0$. So

$$xf(x)^2 - (1-x)f(x) + 1 = 0.$$

Then

$$x\left(\sum_{n=0}^{\infty}a_nx^n\right)^2 + 1 = \sum_{n=0}^{\infty}a_nx^n - \sum_{n=0}^{\infty}a_nx^{n+1} \quad \Rightarrow \quad \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_ka_{n-k}\right)x^{n+1} = \sum_{n=0}^{\infty}(a_{n+1}-a_n)x^{n+1}$$

and from here, we get

$$a_{n+1} = a_n + \sum_{k=0}^{n}a_ka_{n-k}.$$

If $a_0, a_1, ..., a_n$ be integers, then a_{n+1} is also integer. We know that $a_0 = 1, a_1 = 2$ are integer numbers, so all terms of the sequence (a_n) are integers by induction.

Problem 5. Determine whether there exist an odd positive integer n and $n \times n$ matrices A and B with integer entries, that satisfy the following conditions:

(1) $\det(B) = 1$;
(2) $AB = BA$;
(3) $A^4 + 4A^2B^2 + 16B^4 = 2019I$.

(Here I denotes the $n \times n$ identity matrix.)

Proposed by Orif Ibrogimov, ETH Zurich and National University of Uzbekistan

Hint: Consider the determinants modulo 4.

Remark. The proposed solution was more complicated and involved; during the contest it turned out that a signficantly simplified solution exists – which we now provide below.

Solution 1. We show that there are no such matrices.

Notice that $A^4 + 4A^2B^2 + 16B^4$ can factorized as

$$A^4 + 4A^2B^2 + 16B^4 = (A^2 + 2AB + 4B^2)(A^2 - 2AB + 4B^2).$$

Let $C = A^2 + 2AB + 4B^2$ and $D = A^2 - 2AB + 4B^2$ be the two factors above. Then

$$\det C \cdot \det D = \det(CD) = \det(A^4 + 4A^2B^2 + 16B^4) = \det(2019I) = 2019^n.$$

The matrices C, D have integer entries, so their determinants are integers. Moreover, from $C \equiv D \pmod 4$ we can see that

$$\det C \equiv \det D \pmod 4.$$

This implies that $\det C \cdot \det D \equiv (\det C)^2 \pmod 4$, but this is a contradiction because $2019^n \equiv 3 \pmod 4$ is a quadratic nonresidue modulo 4.

Solution 2. Notice that

$$A^4 \equiv A^4 + 4A^2B^2 + 16B^4 = 2019I \mod 4$$

so

$$(\det A)^4 = \det A^4 \equiv \det(2109I) = 2019^n \pmod 4.$$

But $2019^n \equiv 3$ is a quadratic nonresidue modulo 4, contradiction.