Problem 1. Let \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) be two sequences of positive numbers. Show that the following statements are equivalent:

(1) There is a sequence \((c_n)_{n=1}^{\infty}\) of positive numbers such that \(\sum_{n=1}^{\infty} \frac{a_n}{c_n}\) and \(\sum_{n=1}^{\infty} \frac{c_n}{b_n}\) both converge;

(2) \(\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}\) converges.

(Proposed by Tomáš Bárá, Charles University, Prague)

Solution. Note that the sum of a series with positive terms can be either finite or \(+\infty\), so for such a series, "converges" is equivalent to "is finite".

Proof for (1) \(\Rightarrow\) (2): By the AM-GM inequality,

\[
\sqrt{\frac{a_n}{b_n}} = \sqrt{\frac{a_n}{c_n} \cdot \frac{c_n}{b_n}} \leq \frac{1}{2} \left( \frac{a_n}{c_n} + \frac{c_n}{b_n} \right),
\]

so

\[
\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}} \leq \frac{1}{2} \sum_{n=1}^{\infty} \frac{a_n}{c_n} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{c_n}{b_n} < +\infty.
\]

Hence, \(\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}\) is finite and therefore convergent.

Proof for (2) \(\Rightarrow\) (1): Choose \(c_n = \sqrt{a_nb_n}\). Then

\[
\frac{a_n}{c_n} = \frac{c_n}{b_n} = \sqrt{\frac{a_n}{b_n}}.
\]

By the condition \(\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}\) converges, therefore \(\sum_{n=1}^{\infty} \frac{a_n}{c_n}\) and \(\sum_{n=1}^{\infty} \frac{c_n}{b_n}\) converge, too.

Problem 2. Does there exist a field such that its multiplicative group is isomorphic to its additive group?

(Proposed by Alexandre Chapovalov, New York University, Abu Dhabi)

Solution. There exist no such field.

Suppose that \(F\) is such a field and \(g: F^* \rightarrow F^+\) is a group isomorphism. Then \(g(1) = 0\).

Let \(a = g(-1)\). Then \(2a = 2 \cdot g(-1) = g((-1)^2) = g(1) = 0\); so either \(a = 0\) or \(\text{char } F = 2\).

If \(a = 0\) then \(-1 = g^{-1}(a) = g^{-1}(0) = 1\); we have \(\text{char } F = 2\) in any case.

For every \(x \in F\), we have \(g(x^2) = 2g(x) = 0 = g(1)\), so \(x^2 = 1\). But this equation has only one or two solutions. Hence \(F\) is the 2-element field; but its additive and multiplicative groups have different numbers of elements and are not isomorphic.
Problem 3. Determine all rational numbers \(a\) for which the matrix
\[
\begin{pmatrix}
a & -a & -1 & 0 \\
a & -a & 0 & -1 \\
1 & 0 & a & -a \\
0 & 1 & a & -a \\
\end{pmatrix}
\]
is the square of a matrix with all rational entries.
(Proposed by Daniël Kroes, University of California, San Diego)

Solution. We will show that the only such number is \(a = 0\).

Let \(A = \begin{pmatrix} a & -a & -1 & 0 \\ a & -a & 0 & -1 \\ 1 & 0 & a & -a \\ 0 & 1 & a & -a \end{pmatrix}\) and suppose that \(A = B^2\). It is easy to compute the characteristic polynomial of \(A\), which is
\[
p_A(x) = \det(A - xI) = (x^2 + 1)^2.
\]
By the Cayley-Hamilton theorem we have \(p_A(B^2) = p_A(A) = 0\).

Let \(\mu_B(x)\) be the minimal polynomial of \(B\). The minimal polynomial divides all polynomials that vanish at \(B\); in particular \(\mu_B(x)\) must be a divisor of the polynomial \(p_A(x^2) = (x^4 + 1)^2\). The polynomial \(\mu_B(x)\) has rational coefficients and degree at most 4. On the other hand, the polynomial \(x^4 + 1\), being the 8th cyclotomic polynomial, is irreducible in \(\mathbb{Q}[x]\). Hence the only possibility for \(\mu_B\) is \(\mu_B(x) = x^4 + 1\). Therefore,
\[
A^2 + I = \mu_B(B) = 0. \tag{1}
\]
Since we have
\[
A^2 + I = \begin{pmatrix} 0 & 0 & -2a & 2a \\ 0 & 0 & -2a & 2a \\ 2a & -2a & 0 & 0 \\ 2a & -2a & 0 & 0 \end{pmatrix},
\]
the relation (1) forces \(a = 0\).

In case \(a = 0\) we have
\[
A = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \end{pmatrix}^2,
\]
hence \(a = 0\) satisfies the condition.

Problem 4. Find all differentiable functions \(f : (0, \infty) \to \mathbb{R}\) such that
\[
f(b) - f(a) = (b - a)f'(\sqrt{ab}) \quad \text{for all} \quad a, b > 0. \tag{2}
\]
(Proposed by Orif Ibrogimov, National University of Uzbekistan)

Solution. First we show that \(f\) is infinitely many times differentiable. By substituting \(a = \frac{1}{2}t\) and \(b = 2t\) in (2),
\[
f'(t) = \frac{f(2t) - f(\frac{1}{2}t)}{2t^2}. \tag{3}
\]
Consequently, if $f$ is $k$ times differentiable then the right-hand side of (3) is $k$ times differentiable, so the $f'(t)$ on the left-hand-side is $k$ times differentiable as well; hence $f$ is $k + 1$ times differentiable.

Now substitute $b = e^h t$ and $a = e^{-h} t$ in (2), differentiate three times with respect to $h$ then take limits with $h \to 0$:

$$
\left( \frac{\partial}{\partial h} \right)^3 \left( f(e^h t) - f(e^{-h} t) \right) = 0
$$

$$
e^{3h} t^3 f'''(e^h t) + 3e^{2h} t^2 f''(e^h t) + e^{-3h} t^3 f'''(e^{-h} t) + 3e^{-2h} t^2 f''(e^{-h} t) + e^{-h} t f'(e^{-h} t) -
\left( e^h t + e^{-h} t \right) f'(t) = 0
$$

$$
2t^3 f'''(t) + 6t^2 f''(t) = 0
$$

$$
t f'''(t) + 3f''(t) = 0
$$

$$(tf(t))'' = 0.
$$

Consequently, $tf(t)$ is an at most quadratic polynomial of $t$, and therefore

$$
f(t) = C_1 t + \frac{C_2}{t} + C_3
$$

with some constants $C_1, C_2$ and $C_3$.

It is easy to verify that all functions of the form (4) satisfy the equation (1).

**Problem 5.** Let $p$ and $q$ be prime numbers with $p < q$. Suppose that in a convex polygon $P_1 P_2 \ldots P_{pq}$ all angles are equal and the side lengths are distinct positive integers. Prove that

$$
P_1 P_2 + P_2 P_3 + \cdots + P_k P_{k+1} \geq \frac{k^3 + k}{2}
$$

holds for every integer $k$ with $1 \leq k \leq p$.

(Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin)

**Solution.** Place the polygon in the complex plane counterclockwise, so that $P_2 - P_1$ is a positive real number. Let $a_i = |P_{i+2} - P_{i+1}|$, which is an integer, and define the polynomial $f(x) = a_{pq-1} x^{pq-1} + \cdots + a_1 x + a_0$. Let $\omega = e^{2\pi i/p}$; then $P_{i+1} - P_i = a_i - \omega^i$, so $f(\omega) = 0$.

The minimal polynomial of $\omega$ over $\mathbb{Q}[x]$ is the cyclotomic polynomial $\Phi_{pq}(x) = \frac{x^{pq} - 1}{x^p - 1}$, so $\Phi_{pq}(x)$ divides $f(x)$. At the same time, $\Phi_{pq}(x)$ is the greatest common divisor of $s(x) = \frac{x^{pq} - 1}{x^p - 1} = \Phi_q(x^p)$ and $t(x) = \frac{x^p - 1}{x^q - 1} = \Phi_p(x^q)$, so by Bézout’s identity (for real polynomials), we can write $f(x) = s(x) u(x) + t(x) v(x)$, with some polynomials $u(x), v(x)$. These polynomials can be replaced by $u^*(x) = u(x) + w(x) \frac{x^p - 1}{x^q - 1}$ and $v^*(x) = v(x) - w(x) \frac{x^q - 1}{x^p - 1}$, so without loss of generality we may assume that $\deg u \leq p - 1$. Since $\deg a = pq - 1$, this forces $\deg v \leq q - 1$.

Let $u(x) = u_{pq-1} x^{pq-1} + \cdots + u_1 x + u_0$ and $v(x) = v_{pq-1} x^{pq-1} + \cdots + v_1 x + v_0$. Denote by $(i, j)$ the unique integer $n \in \{0, 1, \ldots, pq - 1\}$ with $n \equiv i \pmod{p}$ and $n \equiv j \pmod{q}$. By the choice of $s$ and $t$, we have $a_{(i,j)} = u_i + v_j$. Then

$$
P_1 P_2 + \cdots + P_k P_{k+1} = \sum_{i=0}^{k-1} a_{(i,j)} = \sum_{i=0}^{k-1} u_i + v_i = \frac{1}{k} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} (u_i + v_j)
$$

$$
= \frac{1}{k} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} a_{(i,j)} \geq \frac{1}{k} \left( 1 + 2 + \cdots + k^2 \right) = \frac{k^3 + k}{2}
$$

where (*) uses the fact that the numbers $(i, j)$ are pairwise different.