IMC 2018, Blagoevgrad, Bulgaria

Day 1, July 24, 2018

Problem 1. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of positive numbers. Show that the following statements are equivalent:

(1) There is a sequence $(c_n)_{n=1}^{\infty}$ of positive numbers such that $\sum_{n=1}^{\infty} \frac{a_n}{c_n}$ and $\sum_{n=1}^{\infty} \frac{c_n}{b_n}$ both converge;

(2) $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ converges.

(10 points)

Problem 2. Does there exist a field such that its multiplicative group is isomorphic to its additive group?

(10 points)

Problem 3. Determine all rational numbers a for which the matrix

$$\begin{pmatrix} a & -a & -1 & 0 \\ a & -a & 0 & -1 \\ 1 & 0 & a & -a \\ 0 & 1 & a & -a \end{pmatrix}$$

is the square of a matrix with all rational entries.

(10 points)

Problem 4. Find all differentiable functions $f : (0, \infty) \to \mathbb{R}$ such that

$$f(b) - f(a) = (b - a) f'\left(\sqrt{ab}\right) \quad \text{for all} \quad a, b > 0.$$

(10 points)

Problem 5. Let p and q be prime numbers with $p < q$. Suppose that in a convex polygon $P_1 P_2 \dots P_{pq}$ all angles are equal and the side lengths are distinct positive integers. Prove that

$$P_1P_2 + P_2P_3 + \cdots + P_kP_{k+1} \geq \frac{k^3 + k}{2}$$

holds for every integer k with $1 \leq k \leq p$.

(10 points)