IMC 2017, Blagoevgrad, Bulgaria

Day 1, August 2, 2017

Problem 1. Determine all complex numbers λ for which there exist a positive integer n and a real $n \times n$ matrix A such that $A^2 = A^T$ and λ is an eigenvalue of A.

(10 points)

Problem 2. Let $f\colon \mathbb{R} \to (0, \infty)$ be a differentiable function, and suppose that there exists a constant $L > 0$ such that

$$\left|f'(x) - f'(y)\right| \le L\left|x - y\right|$$

for all x, y. Prove that

$$\left(f'(x)\right)^2 < 2Lf(x)$$

holds for all x.

(10 points)

Problem 3. For any positive integer m, denote by $P\left(m\right)$ the product of positive divisors of m (e.g. $P(6) = 36$). For every positive integer n define the sequence

$$a_1(n) = n, \qquad a_{k+1}(n) = P(a_k(n)) \quad (k = 1, 2, \ldots, 2016).$$

Determine whether for every set $S \subseteq \{1, 2, \ldots, 2017\}$, there exists a positive integer n such that the following condition is satisfied:

For every k with $1 \le k \le 2017$, the number $a_k(n)$ is a perfect square if and only if $k \in S$.

(10 points)

Problem 4. There are n people in a city, and each of them has exactly 1000 friends (friendship is always symmetric). Prove that it is possible to select a group S of people such that at least $n/2017$ persons in S have exactly two friends in S.

(10 points)

Problem 5. Let k and n be positive integers with $n \ge k^2 - 3k + 4$, and let

$$f(z) = z^{n-1} + c_{n-2}z^{n-2} + \ldots + c_0$$

be a polynomial with complex coefficients such that

$$c_0c_{n-2} = c_1c_{n-3} = \ldots = c_{n-2}c_0 = 0.$$

Prove that $f(z)$ and $z^n - 1$ have at most $n - k$ common roots.

(10 points)