IMC 2016, Blagoevgrad, Bulgaria

Day 1, July 27, 2016

Problem 1. Let $f\colon [a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose that f has infinitely many zeros, but there is no $x\in(a,b)$ with $f(x)=f'(x)=0$.

(a) Prove that $f(a)f(b)=0$.

(b) Give an example of such a function on $[0,1]$.

(Proposed by Alexandr Bolbot, Novosibirsk State University)

Solution. (a) Choose a convergent sequence z_n of zeros and let $c=\lim z_n\in[a,b]$. By the continuity of f we obtain $f(c)=0$. We want to show that either $c=a$ or $c=b$, so $f(a)=0$ or $f(b)=0$; then the statement follows.

If c was an interior point then we would have $f(c)=0$ and $f'(c)=\lim\dfrac{f(z_n)-f(c)}{z_n-c}=\lim\dfrac{0-0}{z_n-c}=0$ simultaneously, contradicting the conditions. Hence, $c=a$ or $c=b$.

(b) Let

$$f(x)=\begin{cases} x\sin\dfrac{1}{x} & \text{if } 0<x\le 1\\ 0 & \text{if } x=0.\end{cases}$$

This function has zeros at the points $\dfrac{1}{k\pi}$ for $k=1,2,\dots$, and it is continuous at 0 as well.

In $(0,1)$ we have

$$f'(x)=\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x}.$$

Since $\sin\frac{1}{x}$ and $\cos\frac{1}{x}$ cannot vanish at the same point, we have either $f(x)\ne 0$ or $f'(x)\ne 0$ everywhere in $(0,1)$.

Problem 2. Let k and n be positive integers. A sequence $(A_1,\dots,A_k)$ of $n\times n$ real matrices is *preferred* by Ivan the Confessor if $A_i^2\ne 0$ for $1\le i\le k$, but $A_iA_j=0$ for $1\le i,j\le k$ with $i\ne j$. Show that $k\le n$ in all preferred sequences, and give an example of a preferred sequence with $k=n$ for each n.

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution 1. For every $i=1,\dots,n$, since $A_i\cdot A_i\ne 0$, there is a column $v_i\in\mathbb{R}^n$ in A_i such that $A_iv_i\ne 0$. We will show that the vectors $v_1,\dots,v_k$ are linearly independent; this immediately proves $k\le n$.

Suppose that a linear combination of $v_1,\dots,v_k$ vanishes:

$$c_1v_1+\ldots+c_kv_k=0,\quad c_1,\dots,c_k\in\mathbb{R}.$$

For $i\ne j$ we have $A_iA_j=0$; in particular, $A_iv_j=0$. Now, for each $i=1,\dots,n$, from

$$0=A_i(c_1v_1+\ldots+c_kv_k)=\sum_{j=1}^{k}c_j(A_iv_j)=c_i(A_iv_i)$$

we can see that $c_i=0$. Hence, $c_1=\ldots=c_k=0$.

The case $k=n$ is possible: if A_i has a single 1 in the main diagonal at the ith position and its other entries are zero then $A_i^2=A_i$ and $A_iA_j=0$ for $i\ne j$.

Remark. The solution above can be re-formulated using block matrices in the following way. Consider

$$\begin{pmatrix} A_1 & A_2 & \dots & A_k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{pmatrix} = \begin{pmatrix} A_1^2 & 0 & \dots & 0 \\ 0 & A_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_k^2 \end{pmatrix}.$$

It is easy to see that the rank of the left-hand side is at most n; the rank of the right-hand side is at least k.

Solution 2. Let U_i and K_i be the image and the kernel of the matrix A_i (considered as a linear operator on $\mathbb{R}^n$), respectively. For every pair i, j of indices, we have $U_j \subset K_i$ if and only if $i \neq j$.

Let $X_0 = \mathbb{R}^n$ and let $X_i = K_1 \cap K_2 \cap \dots \cap K_i$ for $i = 1, \dots, k$, so $X_0 \supset X_1 \supset \dots \supset X_k$. Notice also that $U_i \subset X_{i-1}$ because $U_i \subset K_j$ for every $j < i$, and $U_i \not\subset X_i$ because $U_i \not\subset K_i$. Hence, $X_i \neq X_{i-1}$; X_i is a proper subspace of X_{i-1}.

Now, from

$$n = \dim X_0 > \dim X_1 > \dots > \dim X_k \geq 0$$

we get $k \geq n$.

Problem 3. Let n be a positive integer. Also let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers such that $a_i + b_i > 0$ for $i = 1, 2, \dots, n$. Prove that

$$\sum_{i=1}^{n} \frac{a_i b_i - b_i^2}{a_i + b_i} \leq \frac{\sum\limits_{i=1}^{n} a_i \cdot \sum\limits_{i=1}^{n} b_i - \left(\sum\limits_{i=1}^{n} b_i\right)^2}{\sum\limits_{i=1}^{n} (a_i + b_i)}.$$

(Proposed by Daniel Strzelecki, Nicolaus Copernicus University in TorÃžn, Poland)

Solution. By applying the identity

$$\frac{XY - Y^2}{X + Y} = Y - \frac{2Y^2}{X + Y}$$

with $X = a_i$ and $Y = b_i$ to the terms in the LHS and $X = \sum\limits_{i=1}^{n} a_i$ and $Y = \sum\limits_{i=1}^{n} b_i$ to the RHS,

$$LHS = \sum_{i=1}^{n} \frac{a_i b_i - b_i^2}{a_i + b_i} = \sum_{i=1}^{n} \left(b_i - \frac{2b_i^2}{a_i + b_i} \right) = \sum_{i=1}^{n} b_i - 2 \sum_{i=1}^{n} \frac{b_i^2}{a_i + b_i},$$

$$RHS = \frac{\sum\limits_{i=1}^{n} a_i \cdot \sum\limits_{i=1}^{n} b_i - \left(\sum\limits_{i=1}^{n} b_i\right)^2}{\sum\limits_{i=1}^{n} a_i + \sum\limits_{i=1}^{n} b_i} = \sum_{i=1}^{n} b_i - 2 \frac{\left(\sum\limits_{i=1}^{n} b_i\right)^2}{\sum\limits_{i=1}^{n} (a_i + b_i)}.$$

Therefore, the statement is equivalent with

$$\sum_{i=1}^{n} \frac{b_i^2}{a_i + b_i} \geq \frac{\left(\sum\limits_{i=1}^{n} b_i\right)^2}{\sum\limits_{i=1}^{n} (a_i + b_i)},$$

which is the same as the well-known variant of the Cauchy-Schwarz inequality,

$$\sum_{i=1}^{n} \frac{X_i^2}{Y_i} \geq \frac{(X_1 + \dots + X_n)^2}{Y_1 + \dots + Y_n} \quad (Y_1, \dots, Y_n > 0)$$

with $X_i = b_i$ and $Y_i = a_i + b_i$.

Problem 4. Let $n \geq k$ be positive integers, and let $\mathcal{F}$ be a family of finite sets with the following properties:

(i) $\mathcal{F}$ contains at least $\binom{n}{k} + 1$ distinct sets containing exactly k elements;

(ii) for any two sets $A, B \in \mathcal{F}$, their union $A \cup B$ also belongs to $\mathcal{F}$.

Prove that $\mathcal{F}$ contains at least three sets with at least n elements.

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution 1. If $n = k$ then we have at least two distinct sets in the family with exactly n elements and their union, so the statement is true. From now on we assume that $n > k$.

Fix $\binom{n}{k} + 1$ sets of size k in $\mathcal{F}$, call them 'generators'. Let $V \in \mathcal{F}$ be the union of the generators. Since V has at least $\binom{n}{k} + 1$ subsets of size k, we have $|V| > n$.

Call an element $v \in V$ 'appropriate' if v belongs to at most $\binom{n-1}{k-1}$ generators. Then there exist at least $\binom{n}{k} + 1 - \binom{n-1}{k-1} = \binom{n-1}{k} + 1$ generators not containing v. Their union contains at least n elements, and the union does not contain v.

Now we claim that among any n elements $x_1, \ldots, x_n$ of V, there exists an appropriate element. Consider all pairs (G, x_i) such that G is a generator and $x_i \in G$. Every generator has exactly k elements, so the number of such pairs is at most $\left(\binom{n}{k} + 1\right) \cdot k$. If some x_i is not appropriate then x_i is contained in at least $\binom{n-1}{k-1} + 1$ generators; if none of $x_1, \ldots, x_n$ was appropriate then we wold have at least $n \cdot \left(\binom{n-1}{k-1} + 1\right)$ pairs. But $n \cdot \left(\binom{n-1}{k-1} + 1\right) > (\binom{n-1}{k-1} + 1) \cdot k$, so this is not possible; at least one of $x_1, \ldots, x_n$ must be appropriate.

Since $|V| > n$, the set V contains some appropriate element v_1. Let $U_1 \in \mathcal{F}$ be the union of all generators not containing v_1. Then $|U_1| \geq n$ and $v_1 \notin U_1$. Now take an appropriate element v_2 from U_1 and let $U_2 \in \mathcal{F}$ be the union of all generators not containing v_2. Then $|U_2| \geq n$, so we have three sets, V, U_1 and U_2 in $\in \mathcal{F}$ with at least n elements: $V \neq U_1$ because $v_1 \in V$ and $v_1 \notin U_1$, and U_2 is different from V and U_1 because $v_2 \in V, U_1$ but $v_2 \notin U_2$.

Solution 2. We proceed by induction on k, so we can assume that the statement of the problem is known for smaller values of k. By contradiction, assume that $\mathcal{F}$ has less than 3 sets with at least n elements, that is the number of such sets is 0, 1 or 2. We can assume without loss of generality that $\mathcal{F}$ consists of exactly $N := \binom{n}{k} + 1$ distinct sets of size k and all their possible unions. Denote the sets of size k by $S_1, S_2, \ldots$.

Consider a maximal set $I \subset \{1, \ldots, N\}$ such that $A := \bigcup_{i \in I} S_i$ has size less than n, $|A| < n$. This means that adding any S_j for $j \notin I$ makes the size at least n, $|S_j \cup A| \geq n$. First, let's prove that such j exist. Otherwise, all the sets S_i are contained in A. But there are only $\binom{|A|}{k} \leq \binom{n-1}{k} < N$ distinct k-element subsets of A, this is a contradiction. So there is at least one j such that $|S_j \cup A| \geq n$. Consider all possible sets that can be obtained as $S_j \cup A$ for $j \notin I$. Their size is at least n, so their number can be 1 or 2. If there are two of them, say B and C then $B \subset C$ or $C \subset B$, for otherwise the union of B and C would be different from both B and C, so we would have three sets B, C and $B \cup C$ of size at least n. We see that in any case there must exist $x \notin A$ such that $x \in S_j$ for all $j \notin I$. Consider sets $S'_j = S_j \setminus \{x\}$ for $j \notin I$. Their sizes are equal to $k - 1$. Their number is at least

$$N - \binom{n-1}{k} = \binom{n-1}{k-1} + 1.$$

By the induction hypothesis, we can form 3 sets of size at least $n - 1$ by taking unions of the sets S'_j for $j \notin I$. Adding x back we see that the corresponding unions of the sets S_j will have sizes at least n, so we are done proving the induction step.

The above argument allows us to decrease k all the way to $k = 0$, so it remains to check the statement for $k = 0$. The assumption says that we have at least $\binom{n}{0} + 1 = 2$ sets of size 0. This is impossible, because there is only one empty set. Thus the statement trivially holds for $k = 0$.

Problem 5. Let S_n denote the set of permutations of the sequence $(1, 2, \ldots, n)$. For every permutation $\pi = (\pi_1, \ldots, \pi_n) \in S_n$, let $\mathrm{inv}(\pi)$ be the number of pairs $1 \leq i < j \leq n$ with $\pi_i > \pi_j$; i.e. the number of inversions in π. Denote by $f(n)$ the number of permutations $\pi \in S_n$ for which $\mathrm{inv}(\pi)$ is divisible by $n+1$.

Prove that there exist infinitely many primes p such that $f(p-1) > \frac{(p-1)!}{p}$, and infinitely many primes p such that $f(p-1) < \frac{(p-1)!}{p}$.

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution. We will use the well-known formula

$$\sum_{\pi \in S_n} x^{\mathrm{inv}(\pi)} = 1 \cdot (1+x) \cdot (1+x+x^2) \dots (1+x+\cdots+x^{n-1}).$$

(This formula can be proved by induction on n. The cases $n = 1, 2$ are obvious. From each permutation of $(1, 2, \dots, n-1)$, we can get a permutation of $(1, 2, \dots, n)$ such that we insert the element n at one of the n possible positions before, between or after the numbers $1, 2, \dots, n-1$; the number of inversions increases by $n-1, n-2, \dots, 1$ or 0, respectively.)

Now let

$$G_n(x) = \sum_{\pi \in S_n} x^{\mathrm{inv}(\pi)}.$$

and let $\varepsilon = e^{\frac{2\pi i}{n+1}}$. The sum of coefficients of the powers divisible by $n+1$ may be expressed as a trigonometric sum as

$$f(n) = \frac{1}{n+1} \sum_{k=0}^{n-1} G_n(\varepsilon^k) = \frac{n!}{n+1} + \frac{1}{n+1} \sum_{k=1}^{n-1} G_n(\varepsilon^k).$$

Hence, we are interested in the sign of

$$f(n) - \frac{n!}{n+1} = \sum_{k=1}^{n-1} G_n(\varepsilon^k)$$

with $n = p-1$ where p is a (large, odd) prime.

For every fixed $1 \le k \le p-1$ we have

$$G_{p-1}(\varepsilon^k) = \prod_{j=1}^{p-1} (1 + \varepsilon^k + \varepsilon^{2k} + \ldots + \varepsilon^{(j-1)k}) = \prod_{j=1}^{p-1} \frac{1-\varepsilon^{jk}}{1-\varepsilon^k} = \frac{(1-\varepsilon^k)(1-\varepsilon^{2k})\cdots(1-\varepsilon^{(p-1)k})}{(1-\varepsilon^k)^{p-1}}.$$

Notice that the factors in the numerator are $(1-\varepsilon)$, $(1-\varepsilon^2)$, $\dots$, $(1-\varepsilon^{p-1})$; only their order is different. So, by the identity $(z-\varepsilon)(z-\varepsilon^2)\dots(z-\varepsilon^{p-1}) = 1 + z + \cdots + z^{p-1}$,

$$G_{p-1}(\varepsilon^k) = \frac{p}{(1-\varepsilon^k)^{p-1}} = \frac{p}{\left(1 - e^{\frac{2k\pi i}{p}}\right)^{p-1}}.$$

Hence, $f(p-1) - \frac{(p-1)!}{p}$ has the same sign as

$$\sum_{k=1}^{p-1} (1 - e^{\frac{2k\pi i}{p}})^{1-p} = \sum_{k=1}^{p-1} e^{\frac{k(1-p)\pi i}{p}} \left(-2i \sin\frac{\pi k}{p}\right)^{1-p} =$$

$$= 2 \cdot 2^{1-p} (-1)^{\frac{p-1}{2}} \sum_{k=1}^{\frac{p-1}{2}} \cos\frac{\pi k(p-1)}{p} \left(\sin\frac{\pi k}{p}\right)^{1-p}.$$

For large primes p the term with $k = 1$ increases exponentially faster than all other terms, so this term determines the sign of the whole sum. Notice that $\cos\frac{\pi(p-1)}{p}$ converges to -1. So, the sum is positive if $p-1$ is odd and negative if $p-1$ is even. Therefore, for sufficiently large primes, $f(p-1) - \frac{(n-1)!}{p}$ is positive if $p \equiv 3 \pmod 4$ and it is negative if $p \equiv 1 \pmod 4$.