IMC2011, Blagoevgrad, Bulgaria

Day 1, July 30, 2011

Problem 1. Let $f\colon \mathbb{R}\to\mathbb{R}$ be a continuous function. A point x is called a *shadow point* if there exists a point $y\in\mathbb{R}$ with $y>x$ such that $f(y)>f(x)$. Let $a<b$ be real numbers and suppose that

- all the points of the open interval $I=(a,b)$ are shadow points;
- a and b are not shadow points.

Prove that
a) $f(x)\le f(b)$ for all $a<x<b$;
b) $f(a)=f(b)$.

(José Luis Díaz-Barrero, Barcelona)

Solution. (a) We prove by contradiction. Suppose that exists a point $c\in(a,b)$ such that $f(c)>f(b)$.

By Weierstrass' theorem, f has a maximal value m on $[c,b]$; this value is attained at some point $d\in[c,b]$. Since $f(d)=\max\limits_{[c.b]} f\ge f(c)>f(b)$, we have $d\ne b$, so $d\in[c,b)\subset(a,b)$. The point d, lying in (a,b), is a shadow point, therefore $f(y)>f(d)$ for some $y>d$. From combining our inequalities we get $f(y)>f(d)>f(b)$.

Case 1: $y>b$. Then $f(y)>f(b)$ contradicts the assumption that b is not a shadow point.

Case 2: $y\le b$. Then $y\in(d,b]\subset[c,b]$, therefore $f(y)>f(d)=m=\max\limits_{[c,b]} f\ge f(y)$, contradiction again.

(b) Since $a<b$ and a is not a shadow point, we have $f(a)\ge f(b)$.

By part (a), we already have $f(x)\le f(b)$ for all $x\in(a,b)$. By the continuity at a we have

$$f(a)=\lim_{x\to a+0} f(x)\le \lim_{x\to a+0} f(b)=f(b)$$

Hence we have both $f(a)\ge f(b)$ and $f(a)\le f(b)$, so $f(a)=f(b)$.

Problem 2. Does there exist a real 3×3 matrix A such that $\mathrm{tr(A)}=0$ and $A^2+A^t=I$? ($\mathrm{tr(A)}$ denotes the trace of A, A^t is the transpose of A, and I is the identity matrix.)

(Moubinool Omarjee, Paris)

Solution. The answer is NO.

Suppose that $\mathrm{tr(A)}=0$ and $A^2+A^t=I$. Taking the transpose, we have

$$A=I-(A^2)^t=I-(A^t)^2=I-(I-A^2)^2=2A^2-A^4,$$

$$A^4-2A^2+A=0.$$

The roots of the polynomial $x^4-2x^2+x=x(x-1)(x^2+x-1)$ are $0,1,\frac{-1\pm\sqrt5}{2}$ so these numbers can be the eigenvalues of A; the eigenvalues of A^2 can be $0,1,\frac{1\pm\sqrt5}{2}$.

By $\mathrm{tr(A)}=0$, the sum of the eigenvalues is 0, and by $\mathrm{tr(A^2)}=\mathrm{tr(I-A^t)}=3$ the sum of squares of the eigenvalues is 3. It is easy to check that this two conditions cannot be satisfied simultaneously.

Problem 3. Let p be a prime number. Call a positive integer n *interesting* if

$$x^n-1=(x^p-x+1)f(x)+pg(x)$$

for some polynomials f and g with integer coefficients.
a) Prove that the number p^p-1 is interesting.
b) For which p is p^p-1 the minimal interesting number?

(Eugene Goryachko and Fedor Petrov, St. Petersburg)

Solution. (a) Let's reformulate the property of being interesting: n is interesting if x^n-1 is divisible by x^p-x+1 in the ring of polynomials over $\mathbb{F}_p$ (the field of residues modulo p). All further congruences are modulo x^p-x+1 in this ring. We have $x^p\equiv x-1$, then $x^{p^2}=(x^p)^p\equiv(x-1)^p\equiv x^p-1\equiv x-2$, $x^{p^3}=(x^{p^2})^p\equiv(x-2)^p\equiv x^p-2^p\equiv x-2^p-1\equiv x-3$ and so on by Fermat's little theorem, finally $x^{p^p}\equiv x-p\equiv x$,

$$x(x^{p^p-1}-1)\equiv 0.$$

Since the polynomials x^p-x+1 and x are coprime, this implies $x^{p^p-1}-1\equiv 0$.

(b) We write

$$x^{1+p+p^2+\dots+p^{p-1}} = x \cdot x^p \cdot x^{p^2} \cdot \ldots \cdot x^{p^{p-1}} \equiv x(x-1)(x-2)\ldots(x-(p-1)) = x^p - x \equiv -1,$$

hence $x^{2(1+p+p^2+\dots+p^{p-1})} \equiv 1$ and $a = 2(1+p+p^2+\dots+p^{p-1})$ is an interesting number.

If $p > 3$, then $a = \frac{2}{p-1}(p^p-1) < p^p - 1$, so we have an interesting number less than $p^p - 1$. On the other hand, we show that $p = 2$ and $p = 3$ do satisfy the condition. First notice that by $gcd(x^m - 1, x^k - 1) = x^{gcd(m,k)} - 1$, for every fixed p the greatest common divisors of interesting numbers is also an interesting number. Therefore the minimal interesting number divides all interesting numbers. In particular, the minimal interesting number is a divisor of $p^p - 1$.

For $p = 2$ we have $p^p - 1 = 3$, so the minimal interesting number is 1 or 3. But $x^2 - x + 1$ does not divide $x - 1$, so 1 is not interesting. Then the minimal interesting number is 3.

For $p = 3$ we have $p^p - 1 = 26$ whose divisors are $1, 2, 13, 26$. The numbers 1 and 2 are too small and $x^{13} \equiv -1 \not\equiv +1$ as shown above, so none of 1,2 and 13 is interesting. So 26 is the minimal interesting number.

Hence, $p^p - 1$ is the minimal interesting number if and only if $p = 2$ or $p = 3$.

Problem 4. Let $A_1, A_2, \dots, A_n$ be finite, nonempty sets. Define the function

$$f(t) = \sum_{k=1}^{n} \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} (-1)^{k-1} t^{|A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k}|}.$$

Prove that f is nondecreasing on $[0, 1]$.

($|A|$ denotes the number of elements in A.)

(Levon Nurbekyan and Vardan Voskanyan, Yerevan)

Solution 1. Let $\Omega = \bigcup_{i=1}^{n} A_i$. Consider a random subset X of Ω which chosen in the following way: for each $x \in \Omega$, choose the element x for the set X with probability t, independently from the other elements.

Then for any set $C \subset \Omega$, we have

$$P(C \subset X) = t^{|C|}.$$

By the *inclusion-exclusion principle*,

$$P\big((A_1 \subset X) \text{ or } (A_2 \subset X) \text{ or } \ldots \text{ or } (A_n \subset X)\big) =$$

$$= \sum_{k=1}^{n} \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} (-1)^{k-1} P\big(A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k} \subset X\big) =$$

$$= \sum_{k=1}^{n} \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} (-1)^{k-1} t^{|A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_k}|}.$$

The probability $P\big((A_1 \subset X) \text{ or } \ldots \text{ or } (A_n \subset X)\big)$ is a nondecreasing function of the probability t.

Problem 5. Let n be a positive integer and let V be a $(2n-1)$-dimensional vector space over the two-element field. Prove that for arbitrary vectors $v_1, \dots, v_{4n-1} \in V$, there exists a sequence $1 \le i_1 < \ldots < i_{2n} \le 4n - 1$ of indices such that $v_{i_1} + \ldots + v_{i_{2n}} = 0$.

(Ilya Bogdanov, Moscow and Géza Kós, Budapest)

Solution. Let $V = \text{aff}\{v_1, \dots, v_{4n-1}\}$. The statement $v_{i_1} + \dots + v_{i_{2n}} = 0$ is translation-invariant (i.e. replacing the vectors by $v_1 - a, \dots, v_{4n-1} - a$), so we may assume that $0 \in V$. Let $d = \dim V$.

Lemma. The vectors can be permuted in such a way that $v_1 + v_2, v_3 + v_4, \dots, v_{2d-1} + v_{2d}$ form a basis of V.

Proof. We prove by induction on d. If $d = 0$ or $d = 1$ then the statement is trivial.

First choose the vector v_1 such a way that $\text{aff}(v_2, v_3, \dots, v_{4n-1}) = V$; this is possible since V is generated by some $d + 1$ vectors and we have $d + 1 \le 2n < 4n - 1$. Next, choose v_2 such that $v_2 \ne v_1$. (By $d > 0$, not all vectors are the same.)

Now let $\ell = \{0, v_1 + v_2\}$ and let $V' = V/\ell$. For any $w \in V$, let $\tilde{w} = \ell + w = \{w, w + v_1 + v_2\}$ be the class of the factor space V' containing w. Apply the induction hypothesis to the vectors $\tilde{v_3}, \dots, \tilde{v_{4n-1}}$. Since $\dim V' = d - 1$, the vectors can permuted in such a way that $\tilde{v_3} + \tilde{v_4}, \ldots, \tilde{v_{2d-1}} + \tilde{v_{2d}}$ is a basis of V'. Then $v_1 + v_2, v_3 + v_4, \dots, v_{2d-1} + v_{2d}$ is a basis of V.

Now we can assume that $v_1 + v_2, v_3 + v_4, \dots, v_{2d-1} + v_{2d}$ is a basis of V. The vector $w = (v_1 + v_3 + \dots + v_{2d-1}) + (v_{2d+1} + v_{2d+2} + \dots + v_{2n+d})$ is the sum of $2n$ vectors, so $w \in V$. Hence, $w + \varepsilon_1(v_1 + v_2) + \dots + \varepsilon_d(v_{2d-1} + v_{2d}) = 0$ with some $\varepsilon_1, \dots, \varepsilon_d \in \mathbb{F}_2$, therefore

$$\sum_{i=1}^{d} \Big((1 - \varepsilon_i)v_{2i-1} + \varepsilon_i v_{2i}\Big) + \sum_{i=2d+1}^{2n+d} v_i = 0.$$

The left-hand side is the sum of $2n$ vectors.