13^{th} International Mathematics Competition for University Students

Odessa, July 20-26, 2006

Second Day

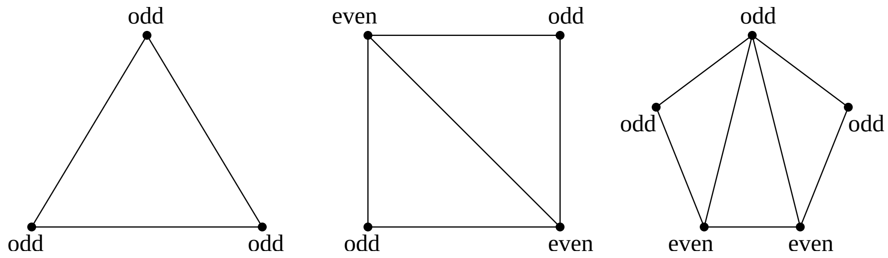
Problem 1. Let V be a convex polygon with n vertices.

(a) Prove that if n is divisible by 3 then V can be triangulated (i.e. dissected into non-overlapping triangles whose vertices are vertices of V) so that each vertex of V is the vertex of an odd number of triangles.

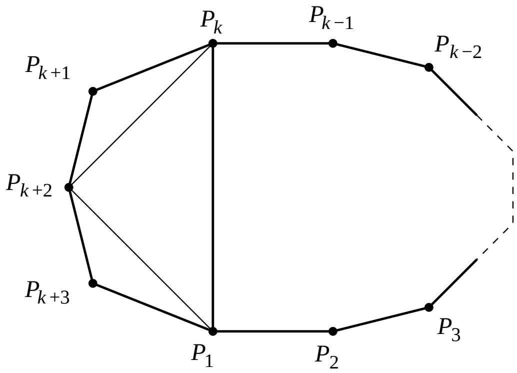
(b) Prove that if n is not divisible by 3 then V can be triangulated so that there are exactly two vertices that are the vertices of an even number of the triangles.

(20 points)

Solution. Apply induction on n. For the initial cases $n = 3, 4, 5$, chose the triangulations shown in the Figure to prove the statement.



Now assume that the statement is true for some $n = k$ and consider the case $n = k + 3$. Denote the vertices of V by $P_1, \ldots, P_{k+3}$. Apply the induction hypothesis on the polygon $P_1P_2 \ldots P_k$; in this triangulation each of vertices $P_1, \ldots, P_k$ belong to an odd number of triangles, except two vertices if n is not divisible by 3. Now add triangles $P_1P_kP_{k+2}$, $P_kP_{k+1}P_{k+2}$ and $P_1P_{k+2}P_{k+3}$. This way we introduce two new triangles at vertices P_1 and P_k so parity is preserved. The vertices P_{k+1}, P_{k+2} and P_{k+3} share an odd number of triangles. Therefore, the number of vertices shared by even number of triangles remains the same as in polygon $P_1P_2 \ldots P_k$.



Problem 2. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for any real numbers $a < b$, the image $f\big([a, b]\big)$ is a closed interval of length $b - a$.

(20 points)

Solution. The functions $f(x) = x + c$ and $f(x) = -x + c$ with some constant c obviously satisfy the condition of the problem. We will prove now that these are the only functions with the desired property.

Let f be such a function. Then f clearly satisfies $|f(x) - f(y)| \leq |x - y|$ for all x, y; therefore, f is continuous. Given x, y with $x < y$, let $a, b \in [x, y]$ be such that $f(a)$ is the maximum and $f(b)$ is the minimum of f on $[x, y]$. Then $f([x, y]) = [f(b), f(a)]$; hence

$$y - x = f(a) - f(b) \leq |a - b| \leq y - x$$

This implies $\{a, b\} = \{x, y\}$, and therefore f is a monotone function. Suppose f is increasing. Then $f(x) - f(y) = x - y$ implies $f(x) - x = f(y) - y$, which says that $f(x) = x + c$ for some constant c. Similarly, the case of a decreasing function f leads to $f(x) = -x + c$ for some constant c.

Problem 3. Compare $\tan(\sin x)$ and $\sin(\tan x)$ for all $x \in (0, \frac{\pi}{2})$.
(20 points)
Solution. Let $f(x) = \tan(\sin x) - \sin(\tan x)$. Then

$$f'(x) = \frac{\cos x}{\cos^2(\sin x)} - \frac{\cos(\tan x)}{\cos^2 x} = \frac{\cos^3 x - \cos(\tan x) \cdot \cos^2(\sin x)}{\cos^2 x \cdot \cos^2(\tan x)}$$

Let $0 < x < \arctan \frac{\pi}{2}$. It follows from the concavity of cosine on $(0, \frac{\pi}{2})$ that

$$\sqrt[3]{\cos(\tan x) \cdot \cos^2(\sin x)} < \frac{1}{3} \left[\cos(\tan x) + 2\cos(\sin x)\right] \leq \cos\left[\frac{\tan x + 2\sin x}{3}\right] < \cos x \ ,$$

the last inequality follows from $\left[\frac{\tan x + 2\sin x}{3}\right]' = \frac{1}{3}\left[\frac{1}{\cos^2 x} + 2\cos x\right] \geq \sqrt[3]{\frac{1}{\cos^2 x} \cdot \cos x \cdot \cos x} = 1$. This proves that $\cos^3 x - \cos(\tan x) \cdot \cos^2(\sin x) > 0$, so $f'(x) > 0$, so f increases on the interval $[0, \arctan \frac{\pi}{2}]$. To end the proof it is enough to notice that (recall that $4 + \pi^2 < 16$)

$$\tan\left[\sin\left(\arctan \frac{\pi}{2}\right)\right] = \tan \frac{\pi/2}{\sqrt{1 + \pi^2/4}} > \tan \frac{\pi}{4} = 1 \ .$$

This implies that if $x \in [\arctan \frac{\pi}{2}, \frac{\pi}{2}]$ then $\tan(\sin x) > 1$ and therefore $f(x) > 0$.

Problem 4. Let v_0 be the zero vector in $\mathbb{R}^n$ and let $v_1, v_2, \ldots, v_{n+1} \in \mathbb{R}^n$ be such that the Euclidean norm $|v_i - v_j|$ is rational for every $0 \leq i, j \leq n+1$. Prove that $v_1, \ldots, v_{n+1}$ are linearly dependent over the rationals.
(20 points)
Solution. By passing to a subspace we can assume that $v_1, \ldots, v_n$ are linearly independent over the reals. Then there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ satisfying

$$v_{n+1} = \sum_{j=1}^{n} \lambda_j v_j$$

We shall prove that λ_j is rational for all j. From

$$-2\langle v_i, v_j \rangle = |v_i - v_j|^2 - |v_i|^2 - |v_j|^2$$

we get that $\langle v_i, v_j \rangle$ is rational for all i, j. Define A to be the rational $n \times n$-matrix $A_{ij} = \langle v_i, v_j \rangle$, $w \in \mathbb{Q}^n$ to be the vector $w_i = \langle v_i, v_{n+1} \rangle$, and $\lambda \in \mathbb{R}^n$ to be the vector $(\lambda_i)_i$. Then,

$$\langle v_i, v_{n+1} \rangle = \sum_{j=1}^{n} \lambda_j \langle v_i, v_j \rangle$$

gives $A\lambda = w$. Since $v_1, \ldots, v_n$ are linearly independent, A is invertible. The entries of A^{-1} are rationals, therefore $\lambda = A^{-1}w \in \mathbb{Q}^n$, and we are done.

Problem 5. Prove that there exists an infinite number of relatively prime pairs (m, n) of positive integers such that the equation

$$(x+m)^3 = nx$$

has three distinct integer roots.
(20 points)

Solution. Substituting $y = x + m$, we can replace the equation by

$$y^3 - ny + mn = 0.$$

Let two roots be u and v; the third one must be $w = -(u+v)$ since the sum is 0. The roots must also satisfy

$$uv + uw + vw = -(u^2 + uv + v^2) = -n, \qquad \text{i.e.} \quad u^2 + uv + v^2 = n$$

and

$$uvw = -uv(u+v) = mn.$$

So we need some integer pairs (u, v) such that $uv(u+v)$ is divisible by $u^2 + uv + v^2$. Look for such pairs in the form $u = kp$, $v = kq$. Then

$$u^2 + uv + v^2 = k^2(p^2 + pq + q^2),$$

and

$$uv(u+v) = k^3 pq(p+q).$$

Chosing p, q such that they are coprime then setting $k = p^2 + pq + q^2$ we have $\dfrac{uv(u+v)}{u^2+uv+v^2} = p^2 + pq + q^2$.

Substituting back to the original quantites, we obtain the family of cases

$$n = (p^2 + pq + q^2)^3, \qquad m = p^2 q + pq^2,$$

and the three roots are

$$x_1 = p^3, \qquad x_2 = q^3, \qquad x_3 = -(p+q)^3.$$

Problem 6. Let A_i, B_i, S_i $(i = 1, 2, 3)$ be invertible real 2×2 matrices such that

(1) not all A_i have a common real eigenvector;

(2) $A_i = S_i^{-1} B_i S_i$ for all $i = 1, 2, 3$;

(3) $A_1 A_2 A_3 = B_1 B_2 B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Prove that there is an invertible real 2×2 matrix S such that $A_i = S^{-1} B_i S$ for all $i = 1, 2, 3$.
(20 points)

Solution. We note that the problem is trivial if $A_j = \lambda I$ for some j, so suppose this is not the case. Consider then first the situation where *some* A_j, say A_3, has two distinct real eigenvalues. We may assume that $A_3 = B_3 = \left(\begin{smallmatrix} \lambda & \\ & \mu \end{smallmatrix}\right)$ by conjugating both sides. Let $A_2 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $B_2 = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$. Then

$$\begin{aligned} a + d = \operatorname{Tr} A_2 &= \operatorname{Tr} B_2 = a' + d' \\ a\lambda + d\mu = \operatorname{Tr}(A_2 A_3) = \operatorname{Tr} A_1^{-1} &= \operatorname{Tr} B_1^{-1} = \operatorname{Tr}(B_2 B_3) = a'\lambda + d'\mu. \end{aligned}$$

Hence $a = a'$ and $d = d'$ and so also $bc = b'c'$. Now we cannot have $c = 0$ or $b = 0$, for then $(1, 0)^\top$ or $(0, 1)^\top$ would be a common eigenvector of all A_j. The matrix $S = \left(\begin{smallmatrix} c' & \\ & c \end{smallmatrix}\right)$ conjugates $A_2 = S^{-1} B_2 S$, and as S commutes with $A_3 = B_3$, it follows that $A_j = S^{-1} B_j S$ for all j.

If the distinct eigenvalues of $A_3 = B_3$ are not real, we know from above that $A_j = S^{-1} B_j S$ for some $S \in \mathrm{GL}_2\mathbb{C}$ unless all A_j have a common eigenvector over $\mathbb{C}$. Even if they do, say $A_j v = \lambda_j v$, by taking the conjugate square root it follows that A_j's can be simultaneously diagonalized. If $A_2 = \left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right)$ and $B_2 = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$, it follows as above that $a = a'$, $d = d'$ and so $b'c' = 0$. Now B_2 and B_3 (and hence B_1 too) have a common eigenvector over $\mathbb{C}$ so they too can be simultaneously diagonalized. And so $SA_j = B_j S$ for some $S \in \mathrm{GL}_2\mathbb{C}$ in either case. Let $S_0 = \operatorname{Re} S$ and $S_1 = \operatorname{Im} S$. By separating the real and imaginary components, we are done if either S_0 or S_1 is invertible. If not, S_0 may be conjugated to some $T^{-1} S_0 T = \left(\begin{smallmatrix} x & 0 \\ y & 0 \end{smallmatrix}\right)$, with $(x, y)^\top \neq (0, 0)^\top$, and it follows that all A_j have a common eigenvector $T(0, 1)^\top$, a contradiction.

We are left with the case when *no* A_j has distinct eigenvalues; then these eigenvalues by necessity are real. By conjugation and division by scalars we may assume that $A_3 = \left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)$ and $b \neq 0$. By further conjugation by upper-triangular matrices (which preserves the shape of A_3 up to the value of b) we can also assume that $A_2 = \left(\begin{smallmatrix} 0 & u \\ 1 & v \end{smallmatrix}\right)$. Here $v^2 = \mathrm{Tr}^2 A_2 = 4 \det A_2 = -4u$. Now $A_1 = A_3^{-1} A_2^{-1} = \left(\begin{smallmatrix} -(b+v)/u & 1 \\ 1/u & \end{smallmatrix}\right)$, and hence $(b + v)^2/u^2 = \mathrm{Tr}^2 A_1 = 4 \det A_1 = -4/u$. Comparing these two it follows that $b = -2v$. What we have done is simultaneously reduced all A_j to matrices whose all entries depend on u and v ($= -\det A_2$ and $\mathrm{Tr}\, A_2$, respectively) only, but these themselves are invariant under similarity. So B_j's can be simultaneously reduced to the very same matrices.