10th **International Mathematical Competition for University Students**
Cluj-Napoca, July 2003

Day 2

1. Let A and B be $n \times n$ real matrices such that $AB + A + B = 0$. Prove that $AB = BA$.

Solution. Since $(A+I)(B+I) = AB+A+B+I = I$ (I is the identity matrix), matrices $A+I$ and $B+I$ are inverses of each other. Then $(A+I)(B+I) = (B+I)(A+I)$ and $AB + BA$.

2. Evaluate the limit

$$\lim_{x\to 0+} \int_x^{2x} \frac{\sin^m t}{t^n} dt \quad (m, n \in \mathbb{N}).$$

Solution. We use the fact that $\dfrac{\sin t}{t}$ is decreasing in the interval $(0,\pi)$ and $\displaystyle\lim_{t\to 0+0} \frac{\sin t}{t} = 1$. For all $x \in (0, \frac{\pi}{2})$ and $t \in [x, 2x]$ we have $\dfrac{\sin 2x}{2}x < \dfrac{\sin t}{t} < 1$, thus

$$\left(\frac{\sin 2x}{2x}\right)^m \int_x^{2x} \frac{t^m}{t^n} < \int_x^{2x} \frac{\sin^m t}{t^n} dt < \int_x^{2x} \frac{t^m}{t^n} dt,$$

$$\int_x^{2x} \frac{t^m}{t^n} dt = x^{m-n+1} \int_1^2 u^{m-n} du.$$

The factor $\left(\dfrac{\sin 2x}{2x}\right)^m$ tends to 1. If $m-n+1 < 0$, the limit of x^{m-n+1} is infinity; if $m-n+1 > 0$ then 0. If $m-n+1 = 0$ then $x^{m-n+1} \int_1^2 u^{m-n} du = \ln 2$. Hence,

$$\lim_{x\to 0+0} \int\limits_x^{2x} \frac{\sin^m t}{t^n} dt = \begin{cases} 0, & m \ge n \\ \ln 2, & n-m = 1 \\ +\infty, & n-m > 1. \end{cases}.$$

3. Let A be a closed subset of $\mathbb{R}^n$ and let B be the set of all those points $b \in \mathbb{R}^n$ for which there exists exactly one point $a_0 \in A$ such that

$$|a_0 - b| = \inf_{a\in A} |a - b|.$$

Prove that B is dense in $\mathbb{R}^n$; that is, the closure of B is $\mathbb{R}^n$.

Solution. Let $b_0 \notin A$ (otherwise $b_0 \in A \subset B$), $\varrho = \inf\limits_{a\in A} |a - b_0|$. The intersection of the ball of radius $\varrho + 1$ with centre b_0 with set A is compact and there exists $a_0 \in A$: $|a_0 - b_0| = \varrho$.

Denote by $\mathbf{B}_r(a) = \{x \in R^n : \ |x-a| \le r\}$ and $\partial\mathbf{B}_r(a) = \{x \in R^n : \ |x-a| = r\}$ the ball and the sphere of center a and radius r, respectively.

If a_0 is not the unique nearest point then for any point a on the open line segment (a_0, b_0) we have $\mathbf{B}_{|a-a_0|}(a) \subset \mathbf{B}_\varrho(b_0)$ and $\partial\mathbf{B}_{|a-a_0|}(a) \bigcap \ \partial\mathbf{B}_\varrho(b_0) = \{a_0\}$, therefore $(a_0, b_0) \subset B$ and b_0 is an accumulation point of set B.

4. Find all positive integers n for which there exists a family $\mathcal{F}$ of three-element subsets of $S = \{1, 2, \ldots, n\}$ satisfying the following two conditions:

(i) for any two different elements $a, b \in S$, there exists exactly one $A \in \mathcal{F}$ containing both a, b;

(ii) if a, b, c, x, y, z are elements of S such that if $\{a, b, x\}, \{a, c, y\}, \{b, c, z\} \in \mathcal{F}$, then $\{x, y, z\} \in \mathcal{F}$.

Solution. The condition (i) of the problem allows us to define a (well-defined) operation $*$ on the set S given by

$$a * b = c \text{ if and only if } \{a, b, c\} \in F, \text{ where } a \ne b.$$

We note that this operation is still not defined completely (we need to define $a * a$), but nevertheless let us investigate its features. At first, due to (i), for $a \ne b$ the operation obviously satisfies the following three conditions:

(a) $a \ne a * b \ne b$;

(b) $a * b = b * a$;

(c) $a * (a * b) = b$.

What does the condition (ii) give? It claims that

(e') $x * (a * c) = x * y = z = b * c = (x * a) * c$

for any three different x, a, c, i.e. that the operation is associative if the arguments are different. Now we can complete the definition of $*$. In order to save associativity for non-different arguments, i.e. to make $b = a * (a * b) = (a * a) * b$ hold, we will add to S an extra element, call it 0, and define

(d) $a * a = 0$ and $a * 0 = 0 * a = a$.

Now it is easy to check that, for any $a, b, c \in S \cup \{0\}$, (a),(b),(c) and (d), still hold, and

(e) $a * b * c := (a * b) * c = a * (b * c)$.

We have thus obtained that $(S \cup \{0\}, *)$ has the structure of a finite Abelian group, whose elements are all of order two. Since the order of every such group is a power of 2, we conclude that $|S \cup \{0\}| = n + 1 = 2^m$ and $n = 2^m - 1$ for some integer $m \ge 1$.

Given $n = 2^m - 1$, according to what we have proven till now, we will construct a family of three-element subsets of S satisfying (i) and (ii). Let us define the operation $*$ in the following manner:

if $a = a_0 + 2a_1 + \ldots + 2^{m-1}a_{m-1}$ and $b = b_0 + 2b_1 + \ldots + 2^{m-1}b_{m-1}$, where a_i, b_i are either 0 or 1, we put $a * b = |a_0 - b_0| + 2|a_1 - b_1| + \ldots + 2^{m-1}|a_{m-1} - b_{m-1}|$.

It is simple to check that this $*$ satisfies (a),(b),(c) and (e'). Therefore, if we include in F all possible triples $a, b, a * b$, the condition (i) follows from (a),(b) and (c), whereas the condition (ii) follows from (e')

The answer is: $n = 2^m - 1$.

5. (a) Show that for each function $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ there exists a function $g : \mathbb{Q} \to \mathbb{R}$ such that $f(x, y) \le g(x) + g(y)$ for all $x, y \in \mathbb{Q}$.

(b) Find a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for which there is no function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) \le g(x) + g(y)$ for all $x, y \in \mathbb{R}$.

Solution. a) Let $\varphi : \mathbb{Q} \to \mathbb{N}$ be a bijection. Define $g(x) = \max\{|f(s,t)| : s, t \in \mathbb{Q}, \varphi(s) \le \varphi(x), \varphi(t) \le \varphi(x)\}$. We have $f(x, y) \le \max\{g(x), g(y)\} \le g(x) + g(y)$.

b) We shall show that the function defined by $f(x, y) = \frac{1}{|x-y|}$ for $x \neq y$ and $f(x, x) = 0$ satisfies the problem. If, by contradiction there exists a function g as above, it results, that $g(y) \ge \frac{1}{|x-y|} - f(x)$ for $x, y \in \mathbb{R},\ x \neq y$; one obtains that for each $x \in \mathbb{R},\ \lim_{y \to x} g(y) = \infty$. We show, that there exists no function g having an infinite limit at each point of a bounded and closed interval $[a, b]$.

For each $k \in \mathbb{N}^+$ denote $A_k = \{x \in [a, b] : |g(x)| \le k\}$.

We have obviously $[a, b] = \cup_{k=1}^{\infty} A_k$. The set $[a, b]$ is uncountable, so at least one of the sets A_k is infinite (in fact uncountable). This set A_k being infinite, there exists a sequence in A_k having distinct terms. This sequence will contain a convergent subsequence $(x_n)_{n \in \mathbb{N}}$ convergent to a point $x \in [a, b]$. But $\lim_{y \to x} g(y) = \infty$ implies that $g(x_n) \to \infty$, a contradiction because $|g(x_n)| \le k,\ \forall n \in \mathbb{N}$.

Second solution for part (b). Let S be the set of all sequences of real numbers. The cardinality of S is $|S| = |\mathbb{R}|^{\aleph_0} = 2^{\aleph_0^2} = 2^{\aleph_0} = |\mathbb{R}|$. Thus, there exists a bijection $h : \mathbb{R} \to S$. Now define the function f in the following way. For any real x and positive integer n, let $f(x, n)$ be the nth element of sequence $h(x)$. If y is not a positive integer then let $f(x, y) = 0$. We prove that this function has the required property.

Let g be an arbitrary $\mathbb{R} \to \mathbb{R}$ function. We show that there exist real numbers x, y such that $f(x, y) > g(x) + g(y)$. Consider the sequence $(n + g(n))_{n=1}^{\infty}$. This sequence is an element of S, thus $(n + g(n))_{n=1}^{\infty} = h(x)$ for a certain real x. Then for an arbitrary positive integer n, $f(x, n)$ is the nth element, $f(x, n) = n + g(n)$. Choosing n such that $n > g(x)$, we obtain $f(x, n) = n + g(n) > g(x) + g(n)$.

6. Let $(a_n)_{n \in \mathbb{N}}$ be the sequence defined by

$$a_0 = 1,\ a_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} \frac{a_k}{n-k+2}.$$

Find the limit

$$\lim_{n \to \infty} \sum_{k=0}^{n} \frac{a_k}{2^k},$$

if it exists.

Solution. Consider the generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$. By induction $0 < a_n \le 1$, thus this series is absolutely convergent for $|x| < 1$, $f(0) = 1$ and the function is positive in the interval $[0, 1)$. The goal is to compute $f(\frac{1}{2})$.

By the recurrence formula,

$$f'(x) = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{a_k}{n-k+2}x^n =$$

$$= \sum_{k=0}^{\infty} a_k x^k \sum_{n=k}^{\infty}\frac{x^{n-k}}{n-k+2} = f(x)\sum_{m=0}^{\infty}\frac{x^m}{m+2}.$$

Then

$$\ln f(x) = \ln f(x) - \ln f(0) = \int_0^x \frac{f'}{f} = \sum_{m=0}^{\infty}\frac{x^{m+1}}{(m+1)(m+2)} =$$

$$= \sum_{m=0}^{\infty}\left(\frac{x^{m+1}}{(m+1)} - \frac{x^{m+1}}{(m+2)}\right) = 1 + \left(1 - \frac{1}{x}\right)\sum_{m=0}^{\infty}\frac{x^{m+1}}{(m+1)} = 1 + \left(1 - \frac{1}{x}\right)\ln\frac{1}{1-x},$$

$$\ln f\left(\frac{1}{2}\right) = 1 - \ln 2,$$

and thus $f(\dfrac{1}{2}) = \dfrac{e}{2}$.