# Solutions for problems in the $9^{th}$ International Mathematics Competition for University Students

Warsaw, July 19 - July 25, 2002

Second Day

**Problem 1.** Compute the determinant of the $n \times n$ matrix $A = [a_{ij}]$,

$$a_{ij} = \begin{cases} (-1)^{|i-j|}, & if \quad i \neq j, \\ 2, & if \quad i = j. \end{cases}$$

*Solution.* Adding the second row to the first one, then adding the third row to the second one, ..., adding the $n$th row to the $(n-1)$th, the determinant does not change and we have

$$\det(A) = \begin{vmatrix} 2 & -1 & +1 & \dots & \pm 1 & \mp 1 \\ -1 & 2 & -1 & \dots & \mp 1 & \pm 1 \\ +1 & -1 & 2 & \dots & \pm 1 & \mp 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mp 1 & \pm 1 & \mp 1 & \dots & 2 & -1 \\ \pm 1 & \mp 1 & \pm 1 & \dots & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ \pm 1 & \mp 1 & \pm 1 & \mp 1 & \dots & -1 & 2 \end{vmatrix}.$$

Now subtract the first column from the second, then subtract the resulting second column from the third, ..., and at last, subtract the $(n-1)$th column from the $n$th column. This way we have

$$det(A) = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & n+1 \end{vmatrix} = n + 1.$$

**Problem 2.** Two hundred students participated in a mathematical contest. They had 6 problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

*Solution.* For each pair of students, consider the set of those problems which was not solved by them. There exist $\binom{200}{2} = 19900$ sets; we have to prove that at least one set is empty.

For each problem, there are at most 80 students who did not solve it. From these students at most $\binom{80}{2} = 3160$ pairs can be selected, so the problem can belong to at most 3160 sets. The 6 problems together can belong to at most $6 \cdot 3160 = 18960$ sets.

Hence, at least $19900 - 18960 = 940$ sets must be empty.

**Problem 3.** For each $n \geq 1$ let

$$a_n = \sum_{k=0}^{\infty} \frac{k^n}{k!}, \quad b_n = \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!}.$$

Show that $a_n \cdot b_n$ is an integer.

*Solution.* We prove by induction on $n$ that $a_n/e$ and $b_n e$ are integers, we prove this for $n = 0$ as well. (For $n = 0$, the term $0^0$ in the definition of the sequences must be replaced by 1.)

From the power series of $e^x$, $a_n = e^1 = e$ and $b_n = e^{-1} = 1/e$.

Suppose that for some $n \geq 0$, $a_0, a_1, \ldots, a_n$ and $b_0, b_1, ...b_n$ are all multipliers of $e$ and $1/e$, respectively. Then, by the binomial theorem,

$$a_{n+1} = \sum_{k=0}^{n} \frac{(k+1)^{n+1}}{(k+1)!} = \sum_{k=0}^{\infty} \frac{(k+1)^n}{k!} = \sum_{k=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{k^m}{k!} =$$

$$= \sum_{m=0}^{n} \binom{n}{m} \sum_{k=0}^{\infty} \frac{k^m}{k!} = \sum_{m=0}^{n} \binom{n}{m} a_m$$

and similarly

$$b_{n+1} = \sum_{k=0}^{n} (-1)^{k+1} \frac{(k+1)^{n+1}}{(k+1)!} = -\sum_{k=0}^{\infty} (-1)^k \frac{(k+1)^n}{k!} =$$

$$= -\sum_{k=0}^{\infty} (-1)^k \sum_{m=0}^{n} \binom{n}{m} \frac{k^m}{k!} = -\sum_{m=0}^{n} \binom{n}{m} \sum_{k=0}^{\infty} (-1)^k \frac{k^m}{k!} = -\sum_{m=0}^{n} \binom{n}{m} b_m.$$

The numbers $a_{n+1}$ and $b_{n+1}$ are expressed as linear combinations of the previous elements with integer coefficients which finishes the proof.

**Problem 4.** In the tetrahedron $OABC$, let $\angle BOC = \alpha$, $\angle COA = \beta$ and $\angle AOB = \gamma$. Let $\sigma$ be the angle between the faces $OAB$ and $OAC$, and let $\tau$ be the angle between the faces $OBA$ and $OBC$. Prove that

$$\gamma > \beta \cdot \cos \sigma + \alpha \cdot \cos \tau.$$

*Solution.* We can assume $OA = OB = OC = 1$. Intersect the unit sphere with center $O$ with the angle domains $AOB$, $BOC$ and $COA$; the intersections are "slices" and their areas are $\frac{1}{2}\gamma$, $\frac{1}{2}\alpha$ and $\frac{1}{2}\beta$, respectively.

Now project the slices $AOC$ and $COB$ to the plane $OAB$. Denote by $C'$ the projection of vertex $C$, and denote by $A'$ and $B'$ the reflections of vertices $A$ and $B$ with center $O$, respectively. By the projection, $OC' < 1$.

The projections of arcs $AC$ and $BC$ are segments of ellipses with long axes $AA'$ and $BB'$, respectively. (The ellipses can be degenerate if $\sigma$ or $\tau$ is right angle.) The two ellipses intersect each other in 4 points; both half ellipses connecting $A$ and $A'$ intersect both half ellipses connecting $B$ and $B'$. There exist no more intersection, because two different conics cannot have more than 4 common points.

The signed areas of the projections of slices $AOC$ and $COB$ are $\frac{1}{2}\alpha \cdot \cos\tau$ and $\frac{1}{2}\beta \cdot \cos\sigma$, respectively. The statement says thet the sum of these signed areas is less than the area of slice $BOA$.

There are three significantly different cases with respect to the signs of $\cos\sigma$ and $\cos\tau$ (see Figure). If both signs are positive (case (a)), then the projections of slices $OAC$ and $OBC$ are subsets of slice $OBC$ without common interior point, and they do not cover the whole slice $OBC$; this implies the statement. In cases (b) and (c) where at least one of the signs is negative, projections with positive sign are subsets of the slice $OBC$, so the statement is obvious again.

**Problem 5.** Let $A$ be an $n \times n$ matrix with complex entries and suppose that $n > 1$. Prove that

$$A\overline{A} = I_n \iff \exists\, S \in GL_n(\mathbb{C}) \quad \text{such that} \quad A = S\overline{S}^{-1}.$$

(If $A = [a_{ij}]$ then $\overline{A} = [\overline{a_{ij}}]$, where $\overline{a_{ij}}$ is the complex conjugate of $a_{ij}$; $GL_n(\mathbb{C})$ denotes the set of all $n \times n$ invertible matrices with complex entries, and $I_n$ is the identity matrix.)

*Solution.* The direction $\Leftarrow$ is trivial, since if $A = S\overline{S}^{-1}$, then $A\overline{A} = S\overline{S}^{-1} \cdot \overline{S}S^{-1} = I_n$.

For the direction $\Rightarrow$, we must prove that there exists an invertible matrix $S$ such that $A\overline{S} = S$.

Let $w$ be an arbitrary complex number which is not 0. Choosing $S = wA + \overline{w}I_n$, we have $A\overline{S} = A(\overline{w}\overline{A} + wI_n) = \overline{w}I_n + wA = S$. If $S$ is singular, then $\frac{1}{w}S = A - (\overline{w}/w)I_n$ is singular as well, so $\overline{w}/w$ is an eigenvalue of $A$. Since $A$ has finitely many eigenvalues and $\overline{w}/w$ can be any complex number on the unit circle, there exist such $w$ that $S$ is invertible.

**Problem 6.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function whose gradient $\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$ exists at every point of $\mathbb{R}^n$ and satisfies the condition

$$\exists L > 0 \quad \forall x_1, x_2 \in \mathbb{R}^n \quad \|\nabla f(x_1) - \nabla f(x_2)\| \le L\|x_1 - x_2\|.$$

Prove that

$$\forall x_1, x_2 \in \mathbb{R}^n \quad \|\nabla f(x_1) - \nabla f(x_2)\|^2 \le L\langle \nabla f(x_1) - \nabla f(x_2), x_1 - x_2\rangle. \quad (1)$$

In this formula $\langle a, b\rangle$ denotes the scalar product of the vectors $a$ and $b$.

*Solution.* Let $g(x) = f(x) - f(x_1) - \langle \nabla f(x_1), x - x_1\rangle$. It is obvious that $g$ has the same properties. Moreover, $g(x_1) = \nabla g(x_1) = 0$ and, due to convexity, $g$ has 0 as the absolute minimum at $x_1$. Next we prove that

$$g(x_2) \ge \frac{1}{2L}\|\nabla g(x_2)\|^2. \quad (2)$$

Let $y_0 = x_2 - \frac{1}{L}\|\nabla g(x_2)\|$ and $y(t) = y_0 + t(x_2 - y_0)$. Then

$$g(x_2) = g(y_0) + \int_0^1 \langle \nabla g(y(t)), x_2 - y_0\rangle\, dt =$$

$$= g(y_0) + \langle \nabla g(x_2), x_2 - y_0\rangle - \int_0^1 \langle \nabla g(x_2) - \nabla g(y(t)), x_2 - y_0\rangle\, dt \ge$$

$$\ge 0 + \frac{1}{L}\|\nabla g(x_2)\|^2 - \int_0^1 \|\nabla g(x_2) - \nabla g(y(t))\| \cdot \|x_2 - y_0\|\, dt \ge$$

$$\ge \frac{1}{L}\|\nabla g(x_2)\|^2 - \|x_2 - y_0\| \int_0^1 L\|x_2 - g(y)\|\, dt =$$

$$= \frac{1}{L}\|\nabla g(x_2)\|^2 - L\|x_2 - y_0\|^2 \int_0^1 t\, dt = \frac{1}{2L}\|\nabla g(x_2)\|^2.$$

Substituting the definition of $g$ into (2), we obtain

$$f(x_2) - f(x_1) - \langle \nabla f(x_1), x_2 - x_1\rangle \ge \frac{1}{2L}\|\nabla f(x_2) - \nabla f(x_1)\|^2,$$

$$\|\nabla f(x_2) - \nabla f(x_1)\|^2 \le 2L\langle \nabla f(x_1), x_1 - x_2\rangle + 2L(f(x_2) - f(x_1)). \quad (3)$$

Exchanging variables $x_1$ and $x_2$, we have

$$\|\nabla f(x_2) - \nabla f(x_1)\|^2 \le 2L\langle \nabla f(x_2), x_2 - x_1\rangle + 2L(f(x_1) - f(x_2)). \quad (4)$$

The statement (1) is the average of (3) and (4).