Solutions for problems in the 9^{th} International Mathematics Competition for University Students

Warsaw, July 19 - July 25, 2002

First Day

Problem 1. A standard parabola is the graph of a quadratic polynomial $y = x^2 + ax + b$ with leading coefficient 1. Three standard parabolas with vertices V_1, V_2, V_3 intersect pairwise at points A_1, A_2, A_3. Let $A \mapsto s(A)$ be the reflection of the plane with respect to the x axis.

Prove that standard parabolas with vertices $s(A_1)$, $s(A_2)$, $s(A_3)$ intersect pairwise at the points $s(V_1)$, $s(V_2)$, $s(V_3)$.

Solution. First we show that the standard parabola with vertex V contains point A if and only if the standard parabola with vertex $s(A)$ contains point $s(V)$.

Let $A = (a, b)$ and $V = (v, w)$. The equation of the standard parabola with vertex $V = (v, w)$ is $y = (x - v)^2 + w$, so it contains point A if and only if $b = (a - v)^2 + w$. Similarly, the equation of the parabola with vertex $s(A) = (a, -b)$ is $y = (x - a)^2 - b$; it contains point $s(V) = (v, -w)$ if and only if $-w = (v - a)^2 - b$. The two conditions are equivalent.

Now assume that the standard parabolas with vertices V_1 and V_2, V_1 and V_3, V_2 and V_3 intersect each other at points A_3, A_2, A_1, respectively. Then, by the statement above, the standard parabolas with vertices $s(A_1)$ and $s(A_2)$, $s(A_1)$ and $s(A_3)$, $s(A_2)$ and $s(A_3)$ intersect each other at points V_3, V_2, V_1, respectively, because they contain these points.

Problem 2. Does there exist a continuously differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{R}$ we have $f(x) > 0$ and $f'(x) = f(f(x))$?

Solution. Assume that there exists such a function. Since $f'(x) = f(f(x)) > 0$, the function is strictly monotone increasing.

By the monotonity, $f(x) > 0$ implies $f(f(x)) > f(0)$ for all x. Thus, $f(0)$ is a lower bound for $f'(x)$, and for all $x < 0$ we have $f(x) < f(0) + x \cdot f(0) = (1 + x)f(0)$. Hence, if $x \le -1$ then $f(x) \le 0$, contradicting the property $f(x) > 0$.

So such function does not exist.

Problem 3. Let n be a positive integer and let

$$a_k = \frac{1}{\binom{n}{k}}, \quad b_k = 2^{k-n}, \quad for \quad k = 1, 2, \ldots, n.$$

Show that

$$\frac{a_1 - b_1}{1} + \frac{a_2 - b_2}{2} + \cdots + \frac{a_n - b_n}{n} = 0. \tag{1}$$

Solution. Since $k\binom{n}{k} = n\binom{n-1}{k-1}$ for all $k \geq 1$, (1) is equivalent to

$$\frac{2^n}{n}\left[\frac{1}{\binom{n-1}{0}} + \frac{1}{\binom{n-1}{1}} + \cdots + \frac{1}{\binom{n-1}{n-1}}\right] = \frac{2^1}{1} + \frac{2^2}{2} + \cdots + \frac{2^n}{n}. \tag{2}$$

We prove (2) by induction. For $n = 1$, both sides are equal to 2.
Assume that (2) holds for some n. Let

$$x_n = \frac{2^n}{n}\left[\frac{1}{\binom{n-1}{0}} + \frac{1}{\binom{n-1}{1}} + \cdots + \frac{1}{\binom{n-1}{n-1}}\right];$$

then

$$x_{n+1} = \frac{2^{n+1}}{n+1}\sum_{k=0}^{n}\frac{1}{\binom{n}{k}} = \frac{2^n}{n+1}\left(1 + \sum_{k=0}^{n-1}\left(\frac{1}{\binom{n}{k}} + \frac{1}{\binom{n}{k+1}}\right) + 1\right) =$$

$$= \frac{2^n}{n+1}\sum_{k=0}^{n-1}\frac{\frac{n-k}{n} + \frac{k+1}{n}}{\binom{n-1}{k}} + \frac{2^{n+1}}{n+1} = \frac{2^n}{n}\sum_{k=0}^{n-1}\frac{1}{\binom{n-1}{k}} + \frac{2^{n+1}}{n+1} = x_n + \frac{2^{n+1}}{n+1}.$$

This implies (2) for $n + 1$.

Problem 4. Let $f\colon [a, b] \to [a, b]$ be a continuous function and let $p \in [a, b]$. Define $p_0 = p$ and $p_{n+1} = f(p_n)$ for $n = 0, 1, 2, \ldots$ Suppose that the set $T_p = \{p_n\colon\ n = 0, 1, 2, \ldots\}$ is closed, i.e., if $x \notin T_p$ then there is a $\delta > 0$ such that for all $x' \in T_p$ we have $|x' - x| \geq \delta$. Show that T_p has finitely many elements.

Solution. If for some $n > m$ the equality $p_m = p_n$ holds then T_p is a finite set. Thus we can assume that all points p_0, p_1, … are distinct. There is a convergent subsequence p_{n_k} and its limit q is in T_p. Since f is continuous $p_{n_k+1} = f(p_{n_k}) \to f(q)$, so all, except for finitely many, points p_n are accumulation points of T_p. Hence we may assume that all of them are accumulation points of T_p. Let $d = \sup\{|p_m - p_n|\colon\quad m, n \geq 0\}$. Let δ_n be positive numbers such that $\sum_{n=0}^{\infty} \delta_n < \frac{d}{2}$. Let I_n be an interval of length less than δ_n centered at p_n such that there are there are infinitely many k's such that $p_k \notin \bigcup_{j=0}^{n} I_j$, this can be done by induction. Let $n_0 = 0$ and n_{m+1} be the smallest integer $k > n_m$ such that $p_k \notin \bigcup_{j=0}^{n_m} I_j$. Since T_p is closed the limit of the subsequence (p_{n_m}) must be in T_p but it is impossible because of the definition of I_n's, of course if the sequence (p_{n_m}) is not convergent we may replace it with its convergent subsequence. The proof is finished.

Remark. If $T_p = \{p_1, p_2, \dots\}$ and each p_n is an accumulation point of T_p, then T_p is the countable union of nowhere dense sets (i.e. the single-element sets $\{p_n\}$). If T is closed then this contradicts the Baire Category Theorem.

Problem 5. Prove or disprove the following statements:
(a) There exists a monotone function $f\colon [0,1] \to [0,1]$ such that for each $y \in [0,1]$ the equation $f(x) = y$ has uncountably many solutions x.
(b) There exists a continuously differentiable function $f\colon [0,1] \to [0,1]$ such that for each $y \in [0,1]$ the equation $f(x) = y$ has uncountably many solutions x.

Solution. a. It does not exist. For each y the set $\{x\colon \quad y = f(x)\}$ is either empty or consists of 1 point or is an interval. These sets are pairwise disjoint, so there are at most countably many of the third type.
b. Let f be such a map. Then for each value y of this map there is an x_0 such that $y = f(x)$ and $f'(x) = 0$, because an uncountable set $\{x\colon \quad y = f(x)\}$ contains an accumulation point x_0 and clearly $f'(x_0) = 0$. For every $\varepsilon > 0$ and every x_0 such that $f'(x_0) = 0$ there exists an open interval I_{x_0} such that if $x \in I_{x_0}$ then $|f'(x)| < \varepsilon$. The union of all these intervals I_{x_0} may be written as a union of pairwise disjoint open intervals J_n. The image of each J_n is an interval (or a point) of length $< \varepsilon \cdot \text{length}(J_n)$ due to Lagrange Mean Value Theorem. Thus the image of the interval $[0,1]$ may be covered with the intervals such that the sum of their lengths is $\varepsilon \cdot 1 = \varepsilon$. This is not possible for $\varepsilon < 1$.

Remarks. 1. The proof of part **b** is essentially the proof of the easy part of A. Sard's theorem about measure of the set of critical values of a smooth map.
2. If only continuity is required, there exists such a function, e.g. the first co-ordinate of the very well known Peano curve which is a continuous map from an interval onto a square.

Problem 6. For an $n\times n$ matrix M with real entries let $\|M\| = \sup\limits_{x\in\mathbb{R}^n\setminus\{0\}} \dfrac{\|Mx\|_2}{\|x\|_2}$, where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^n$. Assume that an $n\times n$ matrix A with real entries satisfies $\|A^k - A^{k-1}\| \le \frac{1}{2002k}$ for all positive integers k. Prove that $\|A^k\| \le 2002$ for all positive integers k.

Solution.

Lemma 1. Let $(a_n)_{n\ge 0}$ be a sequence of non-negative numbers such that $a_{2k}-a_{2k+1} \le a_k^2$, $a_{2k+1}-a_{2k+2} \le a_k a_{k+1}$ for any $k \ge 0$ and $\limsup n a_n < 1/4$. Then $\limsup \sqrt[n]{a_n} < 1$.

Proof. Let $c_l = \sup_{n\ge 2^l}(n+1)a_n$ for $l \ge 0$. We will show that $c_{l+1} \le 4c_l^2$. Indeed, for any integer $n \ge 2^{l+1}$ there exists an integer $k \ge 2^l$ such that $n = 2k$ or $n = 2k+1$. In the first case there is $a_{2k} - a_{2k+1} \le a_k^2 \le \frac{c_l^2}{(k+1)^2} \le \frac{4c_l^2}{2k+1} - \frac{4c_l^2}{2k+2}$, whereas in the second case there is $a_{2k+1} - a_{2k+2} \le a_k a_{k+1} \le \frac{c_l^2}{(k+1)(k+2)} \le \frac{4c_l^2}{2k+2} - \frac{4c_l^2}{2k+3}$.

Hence a sequence $(a_n - \frac{4c_l^2}{n+1})_{n\ge 2^{l+1}}$ is non-decreasing and its terms are non-positive since it converges to zero. Therefore $a_n \le \frac{4c_l^2}{n+1}$ for $n \ge 2^{l+1}$, meaning that $c_{l+1}^2 \le 4c_l^2$. This implies that a sequence $((4c_l)^{2^{-l}})_{l\ge 0}$ is non-increasing and therefore bounded from above by some number $q \in (0,1)$ since all its terms except finitely many are less than 1. Hence $c_l \le q^{2^l}$ for l large enough. For any n between 2^l and 2^{l+1} there is $a_n \le \frac{c_l}{n+1} \le q^{2^l} \le (\sqrt{q})^n$ yielding $\limsup \sqrt[n]{a_n} \le \sqrt{q} < 1$, yielding $\limsup \sqrt[n]{a_n} \le \sqrt{q} < 1$, which ends the proof.

Lemma 2. Let T be a linear map from $\mathbb{R}^n$ into itself. Assume that $\limsup n\|T^{n+1} - T^n\| < 1/4$. Then $\limsup \|T^{n+1} - T^n\|^{1/n} < 1$. In particular T^n converges in the operator norm and T is power bounded.

Proof. Put $a_n = \|T^{n+1} - T^n\|$. Observe that

$$T^{k+m+1} - T^{k+m} = (T^{k+m+2} - T^{k+m+1}) - (T^{k+1} - T^k)(T^{m+1} - T^m)$$

implying that $a_{k+m} \le a_{k+m+1} + a_k a_m$. Therefore the sequence $(a_m)_{m\ge 0}$ satisfies assumptions of Lemma 1 and the assertion of Proposition 1 follows.

Remarks. 1. The theorem proved above holds in the case of an operator T which maps a normed space X into itself, X does not have to be finite dimensional.

2. The constant $1/4$ in Lemma 1 cannot be replaced by any greater number since a sequence $a_n = \frac{1}{4n}$ satisfies the inequality $a_{k+m} - a_{k+m+1} \le a_k a_m$ for any positive integers k and m whereas it does not have exponential decay.

3. The constant $1/4$ in Lemma 2 cannot be replaced by any number greater that $1/e$. Consider an operator $(Tf)(x) = xf(x)$ on $L^2([0,1])$. One can easily

check that $\limsup \|T^{n+1} - T^n\| = 1/e$, whereas T^n does not converge in the operator norm. The question whether in general $\limsup n\|T^{n+1} - T^n\| < \infty$ implies that T is power bounded remains open.

Remark The problem was incorrectly stated during the competition: instead of the inequality $\|A^k - A^{k-1}\| \le \frac{1}{2002k}$, the inequality $\|A^k - A^{k-1}\| \le \frac{1}{2002n}$ was assumed. If $A = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$ then $A^k = \begin{pmatrix} 1 & k\,\varepsilon \\ 0 & 1 \end{pmatrix}$. Therefore $A^k - A^{k-1} = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$, so for sufficiently small ε the condition is satisfied although the sequence $\big(\|A^k\|\big)$ is clearly unbounded.