8^{th} **IMC 2001**
July 19 - July 25
Prague, Czech Republic

Second day

Problem 1.
Let $r, s \geq 1$ be integers and $a_0, a_1, \ldots, a_{r-1}, b_0, b_1, \ldots, b_{s-1}$ be real non-negative numbers such that

$$(a_0 + a_1x + a_2x^2 + \ldots + a_{r-1}x^{r-1} + x^r)(b_0 + b_1x + b_2x^2 + \ldots + b_{s-1}x^{s-1} + x^s) =$$
$$1 + x + x^2 + \ldots + x^{r+s-1} + x^{r+s}.$$

Prove that each a_i and each b_j equals either 0 or 1.

Solution. Multiply the left hand side polynomials. We obtain the following equalities:

$$a_0b_0 = 1, \quad a_0b_1 + a_1b_0 = 1, \quad \ldots$$

Among them one can find equations

$$a_0 + a_1b_{s-1} + a_2b_{s-2} + \ldots = 1$$

and

$$b_0 + b_1a_{r-1} + b_2a_{r-2} + \ldots = 1.$$

From these equations it follows that $a_0, b_0 \leq 1$. Taking into account that $a_0b_0 = 1$ we can see that $a_0 = b_0 = 1$.

Now looking at the following equations we notice that all a's must be less than or equal to 1. The same statement holds for the b's. It follows from $a_0b_1 + a_1b_0 = 1$ that one of the numbers a_1, b_1 equals 0 while the other one must be 1. Follow by induction.

Problem 2.
Let $a_0 = \sqrt{2}$, $b_0 = 2$, $a_{n+1} = \sqrt{2 - \sqrt{4 - a_n^2}}$, $b_{n+1} = \dfrac{2b_n}{2 + \sqrt{4 + b_n^2}}$.

a) Prove that the sequences (a_n), (b_n) are decreasing and converge to 0.

b) Prove that the sequence $(2^n a_n)$ is increasing, the sequence $(2^n b_n)$ is decreasing and that these two sequences converge to the same limit.

c) Prove that there is a positive constant C such that for all n the following inequality holds: $0 < b_n - a_n < \dfrac{C}{8^n}$.

Solution. Obviously $a_2 = \sqrt{2 - \sqrt{2}} < \sqrt{2}$. Since the function $f(x) = \sqrt{2 - \sqrt{4 - x^2}}$ is increasing on the interval $[0, 2]$ the inequality $a_1 > a_2$ implies that $a_2 > a_3$. Simple induction ends the proof of monotonicity of (a_n). In the same way we prove that (b_n) decreases $\left(\text{just notice that } g(x) = \dfrac{2x}{2 + \sqrt{4 + x^2}} = 2/\left(2/x + \sqrt{1 + 4/x^2}\right)\right)$. It is a matter of simple manipulation to prove that $2f(x) > x$ for all $x \in (0, 2)$, this implies that the sequence $(2^n a_n)$ is strictly increasing. The inequality $2g(x) < x$ for $x \in (0,2)$ implies that the sequence $(2^n b_n)$ strictly decreases. By an easy induction one can show that $a_n^2 = \frac{4b_n^2}{4+b_n^2}$ for positive integers n. Since the limit of the *decreasing* sequence $(2^n b_n)$ of *positive* numbers is finite we have

$$\lim 4^n a_n^2 = \lim \frac{4 \cdot 4^n b_n^2}{4+b_n^2} = \lim 4^n b_n^2 \,.$$

We know already that the limits $\lim 2^n a_n$ and $\lim 2^n b_n$ are equal. The first of the two is positive because the sequence $(2^n a_n)$ is strictly increasing. The existence of a number C follows easily from the equalities

$$2^n b_n - 2^n a_n = \big(4^n b_n^2 - \frac{4^{n+1} b_n^2}{4+b_n^2}\big)/\big(2^n b_n + 2^n a_n\big) = \frac{(2^n b_n)^4}{4+b_n^2} \cdot \frac{1}{4^n} \cdot \frac{1}{2^n(b_n+a_n)}$$

and from the existence of positive limits $\lim 2^n b_n$ and $\lim 2^n a_n$.

Remark. The last problem may be solved in a much simpler way by someone who is able to make use of sine and cosine. It is enough to notice that $a_n = 2\sin\dfrac{\pi}{2^{n+1}}$ and $b_n = 2\tan\dfrac{\pi}{2^{n+1}}$.

Problem 3.

Find the maximum number of points on a sphere of radius 1 in $\mathbb{R}^n$ such that the distance between any two of these points is strictly greater than $\sqrt{2}$.

Solution. The unit sphere in $\mathbb{R}^n$ is defined by

$$S_{n-1} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n | \sum_{k=1}^{n} x_k^2 = 1 \right\}.$$

The distance between the points $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ is:

$$d^2(X,Y) = \sum_{k=1}^{n} (x_k - y_k)^2.$$

We have

$$\begin{aligned} d(X,Y) > \sqrt{2} \quad &\Leftrightarrow \quad d^2(X,Y) > 2 \\ &\Leftrightarrow \quad \sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n} y_k^2 + 2\sum_{k=1}^{n} x_k y_k > 2 \\ &\Leftrightarrow \quad \sum_{k=1}^{n} x_k y_k < 0 \end{aligned}$$

Taking account of the symmetry of the sphere, we can suppose that

$$A_1 = (-1, 0, \ldots, 0).$$

For $X = A_1$, $\sum\limits_{k=1}^{n} x_k y_k < 0$ implies $y_1 > 0$, $\forall\, Y \in M_n$.

Let $X = (x_1, \overline{X})$, $Y = (y_1, \overline{Y}) \in M_n \backslash \{A_1\}$, $\overline{X}, \overline{Y} \in \mathbb{R}^{n-1}$.

We have

$$\sum_{k=1}^{n} x_k y_k < 0 \Rightarrow x_1 y_1 + \sum_{k=1}^{n-1} \overline{x}_k \overline{y}_k < 0 \Leftrightarrow \sum_{k=1}^{n-1} x'_k y'_k < 0,$$

where

$$x'_k = \frac{\overline{x}_k}{\sqrt{\sum \overline{x}_k^2}}, \quad y'_k = \frac{\overline{y}_k}{\sqrt{\sum \overline{y}_k^2}}.$$

therefore

$$(x'_1, \dots, x'_{n-1}), (y'_1, \dots, y'_{n-1}) \in S_{n-2}$$

and verifies $\sum\limits_{k=1}^{n} x_k y_k < 0$.

If a_n is the search number of points in $\mathbb{R}^n$ we obtain $a_n \le 1 + a_{n-1}$ and $a_1 = 2$ implies that $a_n \le n+1$.

We show that $a_n = n+1$, giving an example of a set M_n with $(n+1)$ elements satisfying the conditions of the problem.

$$\begin{aligned}
A_1 &= (-1, 0, 0, 0, \dots, 0, 0)\\
A_2 &= \left(\tfrac{1}{n}, -c_1, 0, 0, \dots, 0, 0\right)\\
A_3 &= \left(\tfrac{1}{n}, \tfrac{1}{n-1} \cdot c_1, -c_2, 0, \dots, 0, 0\right)\\
A_4 &= \left(\tfrac{1}{n}, \tfrac{1}{n-1} \cdot c_1, \tfrac{1}{n-1} \cdot c_2, -c_3, \dots, 0, 0\right)\\
A_{n-1} &= \left(\tfrac{1}{n}, \tfrac{1}{n-1} \cdot c_1, \tfrac{1}{n-2} \cdot c_2, \tfrac{1}{n-3} \cdot c_3, \dots, -c_{n-2}, 0\right)\\
A_n &= \left(\tfrac{1}{n}, \tfrac{1}{n-1} \cdot c_1, \tfrac{1}{n-2} \cdot c_1, \tfrac{1}{n-3} \cdot c_3, \dots, \tfrac{1}{2} \cdot c_{n-2}, -c_{n-1}\right)\\
A_{n+1} &= \left(\tfrac{1}{n}, \tfrac{1}{n-1} \cdot c_1, \tfrac{1}{n-2} \cdot c_2, \tfrac{1}{n-3} \cdot c_3, \dots, \tfrac{1}{2} \cdot c_{n-2}, c_{n-1}\right)
\end{aligned}$$

where

$$c_k = \sqrt{\left(1 + \frac{1}{n}\right)\left(1 - \frac{1}{n-k+1}\right)}, \quad k = \overline{1, n-1}.$$

We have $\sum\limits_{k=1}^{n} x_k y_k = -\frac{1}{n} < 0$ and $\sum\limits_{k-=1}^{n} x_k^2 = 1, \quad \forall X, Y \in \{A_1, \dots, A_{n+1}\}$.

These points are on the unit sphere in $\mathbb{R}^n$ and the distance between any two points is equal to

$$d = \sqrt{2}\sqrt{1 + \frac{1}{n}} > \sqrt{2}.$$

Remark. For $n = 2$ the points form an equilateral triangle in the unit circle; for $n = 3$ the four points from a regular tetrahedron and in $\mathbb{R}^n$ the points from an n dimensional regular simplex.

Problem 4.

Let $A = (a_{k,\ell})_{k,\ell=1,\dots,n}$ be an $n \times n$ complex matrix such that for each $m \in \{1, \dots, n\}$ and $1 \le j_1 < \dots < j_m \le n$ the determinant of the matrix $(a_{j_k, j_\ell})_{k,\ell=1,\dots,m}$ is zero. Prove that $A^n = 0$ and that there exists a permutation $\sigma \in S_n$ such that the matrix

$$(a_{\sigma(k),\sigma(\ell)})_{k,\ell=1,\dots,n}$$

has all of its nonzero elements above the diagonal.

Solution. We will only prove (2), since it implies (1). Consider a directed graph G with n vertices $V_1, \dots, V_n$ and a directed edge from V_k to V_ℓ when $a_{k,\ell} \neq 0$. We shall prove that it is acyclic.

Assume that there exists a cycle and take one of minimum length m. Let $j_1 < \dots < j_m$ be the vertices the cycle goes through and let $\sigma_0 \in S_n$ be a permutation such that $a_{j_k, j_{\sigma_0(k)}} \neq 0$ for $k = 1, \dots, m$. Observe that for any other $\sigma \in S_n$ we have $a_{j_k, j_{\sigma(k)}} = 0$ for some $k \in \{1, \dots, m\}$, otherwise we would obtain a different cycle through the same set of vertices and, consequently, a shorter cycle. Finally

$$0 = \det(a_{j_k, j_\ell})_{k,\ell=1,\dots,m}$$

$$= (-1)^{\text{sign } \sigma_0} \prod_{k=1}^{m} a_{j_k, j_{\sigma_0(k)}} + \sum_{\sigma \neq \sigma_0} (-1)^{\text{sign } \sigma} \prod_{k=1}^{m} a_{j_k, j_{\sigma(k)}} \neq 0,$$

which is a contradiction.

Since G is acyclic there exists a topological ordering i.e. a permutation $\sigma \in S_n$ such that $k < \ell$ whenever there is an edge from $V_{\sigma(k)}$ to $V_{\sigma(\ell)}$. It is easy to see that this permutation solves the problem.

Problem 5. Let $\mathbb{R}$ be the set of real numbers. Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) > 0$, and such that

$$f(x+y) \geq f(x) + yf(f(x)) \quad \text{for all}\, x, y \in \mathbb{R}.$$

Solution. Suppose that there exists a function satisfying the inequality. If $f(f(x)) \leq 0$ for all x, then f is a decreasing function in view of the inequalities $f(x+y) \geq f(x) + yf(f(x)) \geq f(x)$ for any $y \leq 0$. Since $f(0) > 0 \geq f(f(x))$, it implies $f(x) > 0$ for all x, which is a contradiction. Hence there is a z such that $f(f(z)) > 0$. Then the inequality $f(z+x) \geq f(z) + xf(f(z))$ shows that $\lim\limits_{x\to\infty} f(x) = +\infty$ and therefore $\lim\limits_{x\to\infty} f(f(x)) = +\infty$. In particular, there exist $x, y > 0$ such that $f(x) \geq 0$, $f(f(x)) > 1$, $y \geq \frac{x+1}{f(f(x))-1}$ and $f(f(x+y+1)) \geq 0$. Then $f(x+y) \geq f(x) + yf(f(x)) \geq x+y+1$ and hence

$$\begin{aligned} f(f(x+y)) &\geq f(x+y+1) + \big(f(x+y) - (x+y+1)\big)f(f(x+y+1)) \geq \\ &\geq f(x+y+1) \geq f(x+y) + f(f(x+y)) \geq \\ &\geq f(x) + yf(f(x)) + f(f(x+y)) > f(f(x+y)). \end{aligned}$$

This contradiction completes the solution of the problem.

Problem 6.
For each positive integer n, let $f_n(\vartheta) = \sin\vartheta \cdot \sin(2\vartheta) \cdot \sin(4\vartheta) \cdots \sin(2^n\vartheta)$.
For all real ϑ and all n, prove that

$$|f_n(\vartheta)| \le \frac{2}{\sqrt{3}}|f_n(\pi/3)|.$$

Solution. We prove that $g(\vartheta) = |\sin\vartheta||\sin(2\vartheta)|^{1/2}$ attains its maximum value $(\sqrt{3}/2)^{3/2}$ at points $2^k\pi/3$ (where k is a positive integer). This can be seen by using derivatives or a classical bound like

$$|g(\vartheta)| = |\sin\vartheta||\sin(2\vartheta)|^{1/2} = \frac{\sqrt{2}}{\sqrt[4]{3}}\left(\sqrt[4]{|\sin\vartheta|\cdot|\sin\vartheta|\cdot|\sin\vartheta|\cdot|\sqrt{3}\cos\vartheta|}\right)^2$$

$$\le \frac{\sqrt{2}}{\sqrt[4]{3}} \cdot \frac{3\sin^2\vartheta + 3\cos^2\vartheta}{4} = \left(\frac{\sqrt{3}}{2}\right)^{3/2}.$$

Hence

$$\left|\frac{f_n(\vartheta)}{f_n(\pi/3)}\right| = \left|\frac{g(\vartheta)\cdot g(2\vartheta)^{1/2}\cdot g(4\vartheta)^{3/4}\cdots g(2^{n-1}\vartheta)^E}{g(\pi/3)\cdot g(2\pi/3)^{1/2}\cdot g(4\pi/3)^{3/4}\cdots g(2^{n-1}\pi/3)^E}\right|\cdot\left|\frac{\sin(2^n\vartheta)}{\sin(2^n\pi/3)}\right|^{1-E/2}$$

$$\le \left|\frac{\sin(2^n\vartheta)}{\sin(2^n\pi/3)}\right|^{1-E/2} \le \left(\frac{1}{\sqrt{3}/2}\right)^{1-E/2} \le \frac{2}{\sqrt{3}}.$$

where $E = \frac{2}{3}(1-(-1/2)^n)$. This is exactly the bound we had to prove.