Solutions for the second day problems at the IMC 2000

Problem 1.

a) Show that the unit square can be partitioned into n smaller squares if n is large enough.

b) Let $d \geq 2$. Show that there is a constant $N(d)$ such that, whenever $n \geq N(d)$, a d-dimensional unit cube can be partitioned into n smaller cubes.

Solution. We start with the following lemma: If a and b be coprime positive integers then every sufficiently large positive integer m can be expressed in the form $ax + by$ with x, y non-negative integers.

Proof of the lemma. The numbers $0, a, 2a, \ldots, (b-1)a$ give a complete residue system modulo b. Consequently, for any m there exists a $0 \leq x \leq b-1$ so that $ax \equiv m \pmod{b}$. If $m \geq (b-1)a$, then $y = (m - ax)/b$, for which $x + by = m$, is a non-negative integer, too.

Now observe that any dissection of a cube into n smaller cubes may be refined to give a dissection into $n + (a^d - 1)$ cubes, for any $a \geq 1$. This refinement is achieved by picking an arbitrary cube in the dissection, and cutting it into a^d smaller cubes. To prove the required result, then, it suffices to exhibit two relatively prime integers of form $a^d - 1$. In the 2-dimensional case, $a_1 = 2$ and $a_2 = 3$ give the coprime numbers $2^2 - 1 = 3$ and $3^2 - 1 = 8$. In the general case, two such integers are $2^d - 1$ and $(2^d - 1)^d - 1$, as is easy to check.

Problem 2. *Let f be continuous and nowhere monotone on $[0,1]$. Show that the set of points on which f attains local minima is dense in $[0,1]$.*

(A function is nowhere monotone if there exists no interval where the function is monotone. A set is dense if each non-empty open interval contains at least one element of the set.)

Solution. Let $(x - \alpha, x + \alpha) \subset [0,1]$ be an arbitrary non-empty open interval. The function f is not monoton in the intervals $[x - \alpha, x]$ and $[x, x + \alpha]$, thus there exist some real numbers $x - \alpha \leq p < q \leq x$, $x \leq r < s \leq x + \alpha$ so that $f(p) > f(q)$ and $f(r) < f(s)$.

By Weierstrass' theorem, f has a global minimum in the interval $[p, s]$. The values $f(p)$ and $f(s)$ are not the minimum, because they are greater than $f(q)$ and $f(s)$, respectively. Thus the minimum is in the interior of the interval, it is a local minimum. So each non-empty interval $(x - \alpha, x + \alpha) \subset [0,1]$ contains at least one local minimum.

Problem 3. *Let $p(z)$ be a polynomial of degree n with complex coefficients. Prove that there exist at least $n + 1$ complex numbers z for which $p(z)$ is 0 or 1.*

Solution. The statement is not true if p is a constant polynomial. We prove it only in the case if n is positive.

For an arbitrary polynomial $q(z)$ and complex number c, denote by $\mu(q, c)$ the largest exponent α for which $q(z)$ is divisible by $(z - c)^\alpha$. (With other words, if c is a root of q, then $\mu(q, c)$ is the root's multiplicity. Otherwise 0.)

Denote by S_0 and S_1 the sets of complex numbers z for which $p(z)$ is 0 or 1, respectively. These sets contain all roots of the polynomials $p(z)$ and $p(z)-1$, thus

$$\sum_{c\in S_0}\mu(p,c)=\sum_{c\in S_1}\mu(p-1,c)=n. \tag{1}$$

The polynomial p' has at most $n-1$ roots ($n>0$ is used here). This implies that

$$\sum_{c\in S_0\cup S_1}\mu(p',c)\le n-1. \tag{2}$$

If $p(c)=0$ or $p(c)-1=0$, then

$$\mu(p,c)-\mu(p'c)=1 \quad\text{or}\quad \mu(p-1,c)-\mu(p'c)=1, \tag{3}$$

respectively. Putting (1), (2) and (3) together we obtain

$$\left|S_0\right|+\left|S_1\right|=\sum_{c\in S_0}\big(\mu(p,c)-\mu(p',c)\big)+\sum_{c\in S_1}\big(\mu(p-1,c)-\mu(p',c)\big)=$$

$$=\sum_{c\in S_0}\mu(p,c)+\sum_{c\in S_1}\mu(p-1,c)-\sum_{c\in S_0\cup S_1}\mu(p',c)\ge n+n-(n-1)=n+1.$$

Problem 4. *Suppose the graph of a polynomial of degree 6 is tangent to a straight line at 3 points A_1, A_2, A_3, where A_2 lies between A_1 and A_3.*

a) Prove that if the lengths of the segments A_1A_2 and A_2A_3 are equal, then the areas of the figures bounded by these segments and the graph of the polynomial are equal as well.

b) Let $k=\dfrac{A_2A_3}{A_1A_2}$, and let K be the ratio of the areas of the appropriate figures. Prove that

$$\frac{2}{7}k^5<K<\frac{7}{2}k^5.$$

Solution. a) Without loss of generality, we can assume that the point A_2 is the origin of system of coordinates. Then the polynomial can be presented in the form

$$y=\big(a_0x^4+a_1x^3+a_2x^2+a_3x+a_4\big)x^2+a_5x,$$

where the equation $y=a_5x$ determines the straight line A_1A_3. The abscissas of the points A_1 and A_3 are $-a$ and a, $a>0$, respectively. Since $-a$ and a are points of tangency, the numbers $-a$ and a must be double roots of the polynomial $a_0x^4+a_1x^3+a_2x^2+a_3x+a_4$. It follows that the polynomial is of the form

$$y=a_0(x^2-a^2)^2+a_5x.$$

The equality follows from the equality of the integrals

$$\int_{-a}^{0} a_0\left(x^2-a^2\right)x^2\mathrm{d}x = \int_{0}^{a} a_0\left(x^2-a^2\right)x^2\mathrm{d}x$$

due to the fact that the function $y = a_0(x^2 - a^2)$ is even.

b) Without loss of generality, we can assume that $a_0 = 1$. Then the function is of the form

$$y = (x+a)^2(x-b)^2x^2 + a_5x,$$

where a and b are positive numbers and $b = ka$, $0 < k < \infty$. The areas of the figures at the segments A_1A_2 and A_2A_3 are equal respectively to

$$\int_{-a}^{0} (x+a)^2(x-b)^2x^2\mathrm{d}x = \frac{a^7}{210}(7k^2+7k+2)$$

and

$$\int_{0}^{b} (x+a)^2(x-b)^2x^2\mathrm{d}x = \frac{a^7}{210}(2k^2+7k+7)$$

Then

$$K = k^5\frac{2k^2+7k+7}{7k^2+7k+2}.$$

The derivative of the function $f(k) = \frac{2k^2+7k+7}{7k^2+7k+2}$ is negative for $0 < k < \infty$. Therefore $f(k)$ decreases from $\frac{7}{2}$ to $\frac{2}{7}$ when k increases from 0 to ∞. Inequalities $\frac{2}{7} < \frac{2k^2+7k+7}{7k^2+7k+2} < \frac{7}{2}$ imply the desired inequalities.

Problem 5. *Let $\mathbb{R}^+$ be the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y \in \mathbb{R}^+$*

$$f(x)f(yf(x)) = f(x+y).$$

First solution. First, if we assume that $f(x) > 1$ for some $x \in \mathbb{R}^+$, setting $y = \dfrac{x}{f(x)-1}$ gives the contradiction $f(x) = 1$. Hence $f(x) \leq 1$ for each $x \in \mathbb{R}^+$, which implies that f is a decreasing function.

If $f(x) = 1$ for some $x \in \mathbb{R}^+$, then $f(x+y) = f(y)$ for each $y \in \mathbb{R}^+$, and by the monotonicity of f it follows that $f \equiv 1$.

Let now $f(x) < 1$ for each $x \in \mathbb{R}^+$. Then f is strictly decreasing function, in particular injective. By the equalities

$$f(x)f(yf(x)) = f(x+y) =$$

$$= f\big(yf(x) + x + y(1 - f(x))\big) = f(yf(x)) f\Big(\big(x + y(1 - f(x))\big) f(yf(x))\Big)$$

we obtain that $x = (x + y(1 - f(x))) f(yf(x))$. Setting $x = 1,\ z = xf(1)$ and $a = \dfrac{1 - f(1)}{f(1)}$, we get $f(z) = \dfrac{1}{1 + az}$.

Combining the two cases, we conclude that $f(x) = \dfrac{1}{1 + ax}$ for each $x \in \mathbb{R}^+$, where $a \geq 0$. Conversely, a direct verification shows that the functions of this form satisfy the initial equality.

Second solution. As in the first solution we get that f is a decreasing function, in particular differentiable almost everywhere. Write the initial equality in the form

$$\frac{f(x + y) - f(x)}{y} = f^2(x) \frac{f(yf(x)) - 1}{yf(x)}.$$

It follows that if f is differentiable at the point $x \in \mathbb{R}^+$, then there exists the limit $\lim\limits_{z \to 0+} \dfrac{f(z) - 1}{z} =: -a$. Therefore $f'(x) = -af^2(x)$ for each $x \in \mathbb{R}^+$, i.e. $\left(\dfrac{1}{f(x)}\right)' = a$, which means that $f(x) = \dfrac{1}{ax + b}$. Substituting in the initial relaton, we find that $b = 1$ and $a \geq 0$.

Problem 6. *For an $m \times m$ real matrix A, e^A is defined as $\sum\limits_{n=0}^{\infty} \frac{1}{n!} A^n$. (The sum is convergent for all matrices.) Prove or disprove, that for all real polynomials p and $m \times m$ real matrices A and B, $p(e^{AB})$ is nilpotent if and only if $p(e^{BA})$ is nilpotent. (A matrix A is nilpotent if $A^k = 0$ for some positive integer k.)*

Solution. First we prove that for any polynomial q and $m \times m$ matrices A and B, the characteristic polinomials of $q(e^{AB})$ and $q(e^{BA})$ are the same. It is easy to check that for any matrix X, $q(e^X) = \sum\limits_{n=0}^{\infty} c_n X^n$ with some real numbers c_n which depend on q. Let

$$C = \sum_{n=1}^{\infty} c_n \cdot (BA)^{n-1} B = \sum_{n=1}^{\infty} c_n \cdot B(AB)^{n-1}.$$

Then $q(e^{AB}) = c_0 I + AC$ and $q(e^{BA}) = c_0 I + CA$. It is well-known that the characteristic polynomials of AC and CA are the same; denote this polynomial by $f(x)$. Then the characteristic polynomials of matrices $q(e^{AB})$ and $q(e^{BA})$ are both $f(x - c_0)$.

Now assume that the matrix $p(e^{AB})$ is nilpotent, i.e. $\big(p(e^{AB})\big)^k = 0$ for some positive integer k. Chose $q = p^k$. The characteristic polynomial of the matrix $q(e^{AB}) = 0$ is x^m, so the same holds for the matrix $q(e^{BA})$. By the theorem of Cayley and Hamilton, this implies that $\big(q(e^{BA})\big)^m = \big(p(e^{BA})\big)^{km} = 0$. Thus the matrix $q(e^{BA})$ is nilpotent, too.