6th INTERNATIONAL COMPETITION FOR UNIVERSITY STUDENTS IN MATHEMATICS

Keszthely, 1999.

Problems and solutions on the second day

1. Suppose that in a not necessarily commutative ring R the square of any element is 0. Prove that $abc + abc = 0$ for any three elements a, b, c. (20 points)

Solution. From $0 = (a+b)^2 = a^2 + b^2 + ab + ba = ab + ba$, we have $ab = -(ba)$ for arbitrary a, b, which implies

$$abc = a(bc) = -\big((bc)\big)a = -\big(b(ca)\big) = (ca)b = c(ab) = -\big((ab)c\big) = -abc.$$

2. We throw a dice (which selects one of the numbers $1, 2, \ldots, 6$ with equal probability) n times. What is the probability that the sum of the values is divisible by 5? (20 points)

Solution 1. For all nonnegative integers n and modulo 5 residue class r, denote by $p_n^{(r)}$ the probability that after n throwing the sum of values is congruent to r modulo n. It is obvious that $p_0^{(0)} = 1$ and $p_0^{(1)} = p_0^{(2)} = p_0^{(3)} = p_0^{(4)} = 0$.

Moreover, for any $n > 0$ we have

$$p_n^{(r)} = \sum_{i=1}^{6} \frac{1}{6} p_{n-1}^{(r-i)}. \tag{1}$$

From this recursion we can compute the probabilities for small values of n and can conjecture that $p_n^{(r)} = \frac{1}{5} + \frac{4}{5 \cdot 6^n}$ if $n \equiv r \pmod 5$ and $p_n^{(r)} = \frac{1}{5} - \frac{1}{5 \cdot 6^n}$ otherwise. From (1), this conjecture can be proved by induction.

Solution 2. Let S be the set of all sequences consisting of digits $1, \ldots, 6$ of length n. We create collections of these sequences.

Let a collection contain sequences of the form

$$\underbrace{66\ldots6}_{k} X Y_1 \ldots Y_{n-k-1},$$

where $X \in \{1, 2, 3, 4, 5\}$ and k and the digits $Y_1, \ldots, Y_{n-k-1}$ are fixed. Then each collection consists of 5 sequences, and the sums of the digits of sequences give a whole residue system mod 5.

Except for the sequence $66\ldots6$, each sequence is the element of one collection. This means that the number of the sequences, which have a sum of digits divisible by 5, is $\frac{1}{5}(6^n - 1) + 1$ if n is divisible by 5, otherwise $\frac{1}{5}(6^n - 1)$.

Thus, the probability is $\frac{1}{5} + \frac{4}{5 \cdot 6^n}$ if n is divisible by 5, otherwise it is $\frac{1}{5} - \frac{1}{5 \cdot 6^n}$.

Solution 3. For arbitrary positive integer k denote by p_k the probability that the sum of values is k. Define the generating function

$$f(x) = \sum_{k=1}^{\infty} p_k x^k = \left(\frac{x + x^2 + x^3 + x^4 + x^5 + x^6}{6} \right)^n.$$

(The last equality can be easily proved by induction.)

Our goal is to compute the sum $\sum\limits_{k=1}^{\infty} p_{5k}$. Let $\varepsilon = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$ be the first 5th root of unity. Then

$$\sum_{k=1}^{\infty} p_{5k} = \frac{f(1) + f(\varepsilon) + f(\varepsilon^2) + f(\varepsilon^3) + f(\varepsilon^4)}{5}.$$

Obviously $f(1)=1$, and $f(\varepsilon^j)=\frac{\varepsilon^{jn}}{6^n}$ for $j=1,2,3,4$. This implies that $f(\varepsilon)+f(\varepsilon^2)+f(\varepsilon^3)+f(\varepsilon^4)$ is $\frac{4}{6^n}$ if n is divisible by 5, otherwise it is $\frac{-1}{6^n}$. Thus, $\sum\limits_{k=1}^{\infty} p_{5k}$ is $\frac{1}{5}+\frac{4}{5\cdot 6^n}$ if n is divisible by 5, otherwise it is $\frac{1}{5}-\frac{1}{5\cdot 6^n}$.

3. Assume that $x_1,\dots,x_n \geq -1$ and $\sum\limits_{i=1}^{n} x_i^3=0$. Prove that $\sum\limits_{i=1}^{n} x_i \leq \frac{n}{3}$. (20 points)

Solution. The inequality

$$0 \leq x^3-\frac{3}{4}x+\frac{1}{4}=(x+1)\left(x-\frac{1}{2}\right)^2$$

holds for $x \geq -1$.

Substituting $x_1,\dots,x_n$, we obtain

$$0 \leq \sum_{i=1}^{n}\left(x_i^3-\frac{3}{4}x_i+\frac{1}{4}\right)=\sum_{i=1}^{n}x_i^3-\frac{3}{4}\sum_{i=1}^{n}x_i+\frac{n}{4}=0-\frac{3}{4}\sum_{i=1}^{n}x_i+\frac{n}{4},$$

so $\sum\limits_{i=1}^{n} x_i \leq \frac{n}{3}$.

Remark. Equailty holds only in the case when $n=9k$, k of the $x_1,\dots,x_n$ are -1, and $8k$ of them are $\frac{1}{2}$.

4. Prove that there exists no function $f:(0,+\infty)\to(0,+\infty)$ such that $f^2(x)\geq f(x+y)\big(f(x)+y\big)$ for any $x,y>0$. (20 points)

Solution. Assume that such a function exists. The initial inequality can be written in the form $f(x)-f(x+y)\geq f(x)-\frac{f^2(x)}{f(x)+y}=\frac{f(x)y}{f(x)+y}$. Obviously, f is a decreasing function. Fix $x>0$ and choose $n\in\mathbf{N}$ such that $nf(x+1)\geq 1$. For $k=0,1,\dots,n-1$ we have

$$f\left(x+\frac{k}{n}\right)-f\left(x+\frac{k+1}{n}\right)\geq\frac{f\big(x+\frac{k}{n}\big)}{nf\big(x+\frac{k}{n}\big)+1}\geq\frac{1}{2n}.$$

The additon of these inequalities gives $f(x+1)\leq f(x)-\frac{1}{2}$. From this it follows that $f(x+2m)\leq f(x)-m$ for all $m\in\mathbf{N}$. Taking $m\geq f(x)$, we get a contradiction with the conditon $f(x)>0$.

5. Let S be the set of all words consisting of the letters x,y,z, and consider an equivalence relation $\sim$ on S satisfying the following conditions: for arbitrary words $u,v,w\in S$

(i) $uu\sim u$;

(ii) if $v\sim w$, then $uv\sim uw$ and $vu\sim wu$.

Show that every word in S is equivalent to a word of length at most 8. (20 points)

Solution. First we prove the following lemma: If a word $u\in S$ contains at least one of each letter, and $v\in S$ is an arbitrary word, then there exists a word $w\in S$ such that $uvw\sim u$.

If v contains a single letter, say x, write u in the form $u=u_1xu_2$, and choose $w=u_2$. Then $uvw=(u_1xu_2)xu_2=u_1((xu_2)(xu_2))\sim u_1(xu_2)=u$.

In the general case, let the letters of v be $a_1,\dots,a_k$. Then one can choose some words $w_1,\dots,w_k$ such that $(ua_1)w_1\sim u$, $(ua_1a_2)w_2\sim ua_1$, $\dots$, $(ua_1\dots a_k)w_k\sim ua_1\dots a_{k-1}$. Then $u\sim ua_1w_1\sim ua_1a_2w_2w_1\sim\dots\sim ua_1\dots a_kw_k\dots w_1=uv(w_k\dots w_1)$, so $w=w_k\dots w_1$ is a good choice.

Consider now an arbitrary word a, which contains more than 8 digits. We shall prove that there is a shorter word which is equivalent to a. If a can be written in the form $uvvw$, its length can be reduced by $uvvw\sim uvw$. So we can assume that a does not have this form.

Write a in the form $a=bcd$, where b and d are the first and last four letter of a, respectively. We prove that $a\sim bd$.

It is easy to check that b and d contains all the three letters x, y and z, otherwise their length could be reduced. By the lemma there is a word e such that $b(cd)e \sim b$, and there is a word f such that $def \sim d$. Then we can write

$$a = bcd \sim bc(def) \sim bc(dedef) = (bcde)(def) \sim bd.$$

Remark. Of course, it is enough to give for every word of length 9 an shortest shorter word. Assuming that the first letter is x and the second is y, it is easy (but a little long) to check that there are 18 words of length 9 which cannot be written in the form $uvvw$.

For five of these words there is a 2-step solution, for example

$$xyxzyzx\,\underline{zy} \sim xy\,\underline{xzyz\,xzyz}\,y \sim xyx\,\underline{zy\,zy} \sim xyxzy.$$

In the remaining 13 cases we need more steps. The general algorithm given by the Solution works for these cases as well, but needs also very long words. For example, to reduce the length of the word $a = xyzyxzxyz$, we have set $b = xyzy$, $c = x$, $d = zxyz$, $e = xyxzxzyxyzy$, $f = zyxyxzyxzxzxzxyxyzxyz$. The longest word in the algorithm was

$$bcdedef = xyzyxzxyzxyxzxzyxyzyzxyzxyxzxzyxyzyzyxyxzyxzxzxzxyxyzxyz,$$

which is of length 46. This is not the shortest way: reducing the length of word a can be done for example by the following steps:

$$xyzyxzx\,\underline{yz} \sim xyzyxz\,\underline{xyzy}\,z \sim xyzyxzxy\,\underline{zyx}\,yzyz \sim \underline{xyzyxz\,xyzyxz}\,yx\,\underline{yz\,yz} \sim xy\,\underline{zyx\,zyx}\,yz \sim xyzyxyz.$$

(The last example is due to Nayden Kambouchev from Sofia University.)

6. Let A be a subset of $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$ containing at most $\frac{1}{100}\ln n$ elements. Define the rth Fourier coefficient of A for $r \in \mathbf{Z}_n$ by

$$f(r) = \sum_{s\in A} \exp\left(\frac{2\pi i}{n} sr\right).$$

Prove that there exists an $r \neq 0$, such that $\left|f(r)\right| \geq \frac{|A|}{2}$. (20 points)

Solution. Let $A = \{a_1, \ldots, a_k\}$. Consider the k-tuples

$$\left(\exp\frac{2\pi i a_1 t}{n}, \ldots, \exp\frac{2\pi i a_k t}{n}\right) \in \mathbf{C}^k, \qquad t = 0, 1, \ldots, n-1.$$

Each component is in the unit circle $|z| = 1$. Split the circle into 6 equal arcs. This induces a decomposition of the k-tuples into 6^k classes. By the condition $k \leq \frac{1}{100}\ln n$ we have $n > 6^k$, so there are two k-tuples in the same class say for $t_1 < t_2$. Set $r = t_2 - t_1$. Then

$$\operatorname{Re}\exp\frac{2\pi i a_j r}{n} = \cos\left(\frac{2\pi a_j t_2}{n} - \frac{2\pi a_j t_1}{n}\right) \geq \cos\frac{\pi}{3} = \frac{1}{2}$$

for all j, so

$$|f(r)| \geq \operatorname{Re}\, f(r) \geq \frac{k}{2}.$$