6th INTERNATIONAL COMPETITION FOR UNIVERSITY STUDENTS IN MATHEMATICS

Keszthely, 1999.

Problems and solutions on the first day

1. a) Show that for any $m \in \mathbf{N}$ there exists a real $m \times m$ matrix A such that $A^3 = A + I$, where I is the $m \times m$ identity matrix. (6 points)
b) Show that $\det A > 0$ for every real $m \times m$ matrix satisfying $A^3 = A + I$. (14 points)

Solution. a) The diagonal matrix

$$A = \lambda I = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$$

is a solution for equation $A^3 = A + I$ if and only if $\lambda^3 = \lambda + 1$, because $A^3 - A - I = (\lambda^3 - \lambda - 1)I$. This equation, being cubic, has real solution.

b) It is easy to check that the polynomial $p(x) = x^3 - x - 1$ has a positive real root λ_1 (because $p(0) < 0$) and two conjugated complex roots λ_2 and λ_3 (one can check the discriminant of the polynomial, which is $\left(\frac{-1}{3}\right)^3 + \left(\frac{-1}{2}\right)^2 = \frac{23}{108} > 0$, or the local minimum and maximum of the polynomial).

If a matrix A satisfies equation $A^3 = A + I$, then its eigenvalues can be only λ_1, λ_2 and λ_3. The multiplicity of λ_2 and λ_3 must be the same, because A is a real matrix and its characteristic polynomial has only real coefficients. Denoting the multiplicity of λ_1 by α and the common multiplicity of λ_2 and λ_3 by β,

$$\det A = \lambda_1^\alpha \lambda_2^\beta \lambda_3^\beta = \lambda_1^\alpha \cdot (\lambda_2\lambda_3)^\beta.$$

Because λ_1 and $\lambda_2\lambda_3 = |\lambda_2|^2$ are positive, the product on the right side has only positive factors.

2. Does there exist a bijective map $\pi : \mathbf{N} \to \mathbf{N}$ such that

$$\sum_{n=1}^{\infty} \frac{\pi(n)}{n^2} < \infty?$$

(20 points)

Solution 1. No. For, let π be a permutation of $\mathbf{N}$ and let $N \in \mathbf{N}$. We shall argue that

$$\sum_{n=N+1}^{3N} \frac{\pi(n)}{n^2} > \frac{1}{9}.$$

In fact, of the $2N$ numbers $\pi(N+1), \ldots, \pi(3N)$ only N can be $\le N$ so that at least N of them are $> N$. Hence

$$\sum_{n=N+1}^{3N} \frac{\pi(n)}{n^2} \ge \frac{1}{(3N)^2} \sum_{n=N+1}^{3N} \pi(n) > \frac{1}{9N^2} \cdot N \cdot N = \frac{1}{9}.$$

Solution 2. Let π be a permutation of $\mathbf{N}$. For any $n \in \mathbf{N}$, the numbers $\pi(1), \ldots, \pi(n)$ are distinct positive integers, thus $\pi(1) + \ldots + \pi(n) \ge 1 + \ldots + n = \frac{n(n+1)}{2}$. By this inequality,

$$\sum_{n=1}^{\infty} \frac{\pi(n)}{n^2} = \sum_{n=1}^{\infty} \left(\pi(1) + \ldots + \pi(n)\right) \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \ge$$

$$\ge \sum_{n=1}^{\infty} \frac{n(n+1)}{2} \cdot \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{2n(n+1)} \ge \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.$$

3. Suppose that a function $f : \mathbf{R} \to \mathbf{R}$ satisfies the inequality

$$\left| \sum_{k=1}^{n} 3^k \big(f(x+ky) - f(x-ky) \big) \right| \le 1 \tag{1}$$

for every positive integer n and for all $x, y \in \mathbf{R}$. Prove that f is a constant function. (20 points)

Solution. Writing (1) with $n-1$ instead of n,

$$\left| \sum_{k=1}^{n-1} 3^k \big(f(x+ky) - f(x-ky) \big) \right| \le 1. \tag{2}$$

From the difference of (1) and (2),

$$\left| 3^n \big(f(x+ny) - f(x-ny) \big) \right| \le 2;$$

which means

$$\left| f(x+ny) - f(x-ny) \right| \le \frac{2}{3^n}. \tag{3}$$

For arbitrary $u, v \in \mathbf{R}$ and $n \in \mathbf{N}$ one can choose x and y such that $x - ny = u$ and $x + ny = v$, namely $x = \frac{u+v}{2}$ and $y = \frac{v-u}{2n}$. Thus, (3) yields

$$\left| f(u) - f(v) \right| \le \frac{2}{3^n}$$

for arbitrary positive integer n. Because $\frac{2}{3^n}$ can be arbitrary small, this implies $f(u) = f(v)$.

4. Find all strictly monotonic functions $f : (0, +\infty) \to (0, +\infty)$ such that $f\big(\frac{x^2}{f(x)}\big) \equiv x$. (20 points)

Solution. Let $g(x) = \dfrac{f(x)}{x}$. We have $g(\dfrac{x}{g(x)}) = g(x)$. By induction it follows that $g(\dfrac{x}{g^n(x)}) = g(x)$, i.e.

$$(1) \qquad f(\frac{x}{g^n(x)}) = \frac{x}{g^{n-1}(x)}, \ n \in \mathbf{N}.$$

On the other hand, let substitute x by $f(x)$ in $f(\dfrac{x^2}{f(x)}) = x$. ¿From the injectivity of f we get $\dfrac{f^2(x)}{f(f(x))} = x$, i.e. $g(xg(x)) = g(x)$. Again by induction we deduce that $g(xg^n(x)) = g(x)$ which can be written in the form

$$(2) \qquad f(xg^n(x)) = xg^{n-1}(x), \ n \in \mathbf{N}.$$

Set $f^{(m)} = \underbrace{f \circ f \circ \ldots \circ f}_{m \ times}$. It follows from (1) and (2) that

$$(3) \qquad f^{(m)}(xg^n(x)) = xg^{n-m}(x), \ m, n \in \mathbf{N}.$$

Now, we shall prove that g is a constant. Assume $g(x_1) < g(x_2)$. Then we may find $n \in \mathbf{N}$ such that $x_1 g^n(x_1) \le x_2 g^n(x_2)$. On the other hand, if m is even then $f^{(m)}$ is strictly increasing and from (3) it follows that $x_1^m g^{n-m}(x_1) \le x_2^m g^{n-m}(x_2)$. But when n is fixed the opposite inequality holds $\forall m \gg 1$. This contradiction shows that g is a constant, i.e. $f(x) = Cx, \ C > 0$.

Conversely, it is easy to check that the functions of this type verify the conditions of the problem.

5. Suppose that $2n$ points of an $n \times n$ grid are marked. Show that for some $k > 1$ one can select $2k$ distinct marked points, say $a_1, \ldots, a_{2k}$, such that a_1 and a_2 are in the same row, a_2 and a_3 are in the same column, $\ldots$, a_{2k-1} and a_{2k} are in the same row, and a_{2k} and a_1 are in the same column. (20 points)

Solution 1. We prove the more general statement that if at least $n+k$ points are marked in an $n \times k$ grid, then the required sequence of marked points can be selected.

If a row or a column contains at most one marked point, delete it. This decreases $n+k$ by 1 and the number of the marked points by at most 1, so the condition remains true. Repeat this step until each row and column contains at least two marked points. Note that the condition implies that there are at least two marked points, so the whole set of marked points cannot be deleted.

We define a sequence $b_1, b_2, \dots$ of marked points. Let b_1 be an arbitrary marked point. For any positive integer n, let b_{2n} be an other marked point in the row of b_{2n-1} and b_{2n+1} be an other marked point in the column of b_{2n}.

Let m be the first index for which b_m is the same as one of the earlier points, say $b_m = b_l$, $l < m$.

If $m-l$ is even, the line segments $b_l b_{l+1}$, $b_{l+1} b_{l+2}$, ..., $b_{m-1} b_l = b_{m-1} b_m$ are alternating horizontal and vertical. So one can choose $2k = m-l$, and $(a_1, \dots, a_{2k}) = (b_l, \dots, b_{m-1})$ or $(a_1, \dots, a_{2k}) = (b_{l+1}, \dots, b_m)$ if l is odd or even, respectively.

If $m-l$ is odd, then the points $b_l = b_m$, b_{l+1} and b_{m-1} are in the same row/column. In this case chose $2k = m-l-1$. Again, the line segments $b_{l+1} b_{l+2}$, $b_{l+2} b_{l+3}$, ..., $b_{m-1} b_{l+1}$ are alternating horizontal and vertical and one can choose $(a_1, \dots, a_{2k}) = (b_{l+1}, \dots, b_{m-1})$ or $(a_1, \dots, a_{2k}) = (b_{l+2}, \dots, b_{m-1}, b_{l+1})$ if l is even or odd, respectively.

Solution 2. Define the graph G in the following way: Let the vertices of G be the rows and the columns of the grid. Connect a row r and a column c with an edge if the intersection point of r and c is marked.

The graph G has $2n$ vertices and $2n$ edges. As is well known, if a graph of N vertices contains no circle, it can have at most $N-1$ edges. Thus G does contain a circle. A circle is an alternating sequence of rows and columns, and the intersection of each neighbouring row and column is a marked point. The required sequence consists of these intersection points.

6. a) For each $1 < p < \infty$ find a constant $c_p < \infty$ for which the following statement holds: If $f : [-1,1] \to \mathbf{R}$ is a continuously differentiable function satisfying $f(1) > f(-1)$ and $|f'(y)| \le 1$ for all $y \in [-1,1]$, then there is an $x \in [-1,1]$ such that $f'(x) > 0$ and $|f(y) - f(x)| \le c_p \big(f'(x)\big)^{1/p} |y-x|$ for all $y \in [-1,1]$. (10 points)
b) Does such a constant also exist for $p = 1$? (10 points)

Solution. (a) Let $g(x) = \max(0, f'(x))$. Then $0 < \int_{-1}^1 f'(x)dx = \int_{-1}^1 g(x)dx + \int_{-1}^1 (f'(x) - g(x))dx$, so we get $\int_{-1}^1 |f'(x)|dx = \int_{-1}^1 g(x)dx + \int_{-1}^1 (g(x) - f'(x))dx < 2\int_{-1}^1 g(x)dx$. Fix p and c (to be determined at the end). Given any $t > 0$, choose for every x such that $g(x) > t$ an interval $I_x = [x,y]$ such that $|f(y) - f(x)| > cg(x)^{1/p}|y-x| > ct^{1/p}|I_x|$ and choose disjoint I_{x_i} that cover at least one third of the measure of the set $\{g > t\}$. For $I = \bigcup_i I_i$ we thus have $ct^{1/p}|I| \le \int_I f'(x)dx \le \int_{-1}^1 |f'(x)|dx < 2\int_{-1}^1 g(x)dx$; so $|\{g > t\}| \le 3|I| < (6/c)t^{-1/p} \int_{-1}^1 g(x)dx$. Integrating the inequality, we get $\int_{-1}^1 g(x)dx = \int_0^1 |\{g > t\}|dt < (6/c)p/(p-1)\int_{-1}^1 g(x)dx$; this is a contradiction e.g. for $c_p = (6p)/(p-1)$.

(b) No. Given $c > 1$, denote $\alpha = 1/c$ and choose $0 < \varepsilon < 1$ such that $((1+\varepsilon)/(2\varepsilon))^{-\alpha} < 1/4$. Let $g : [-1,1] \to [-1,1]$ be continuous, even, $g(x) = -1$ for $|x| \le \varepsilon$ and $0 \le g(x) < \alpha((|x|+\varepsilon)/(2\varepsilon))^{-\alpha-1}$ for $\varepsilon < |x| \le 1$ is chosen such that $\int_\varepsilon^1 g(t)dt > -\varepsilon/2 + \int_\varepsilon^1 \alpha((|x|+\varepsilon)/(2\varepsilon))^{-\alpha-1}dt = -\varepsilon/2 + 2\varepsilon(1 - ((1+\varepsilon)/(2\varepsilon))^{-\alpha}) > \varepsilon$. Let $f = \int g(t)dt$. Then $f(1) - f(-1) \ge -2\varepsilon + 2\int_\varepsilon^1 g(t)dt > 0$. If $\varepsilon < x < 1$ and $y = -\varepsilon$, then $|f(x) - f(y)| \ge 2\varepsilon - \int_\varepsilon^x g(t)dt \ge 2\varepsilon - \int_\varepsilon^x \alpha((t+\varepsilon)/(2\varepsilon))^{-\alpha-1} = 2\varepsilon((x+\varepsilon)/(2\varepsilon))^{-\alpha} > g(x)|x-y|/\alpha = f'(x)|x-y|/\alpha$; symmetrically for $-1 < x < -\varepsilon$ and $y = \varepsilon$.