International Competition in Mathematics for Universtiy Students in Plovdiv, Bulgaria 1994

PROBLEMS AND SOLUTIONS

First day — July 29, 1994

Problem 1. (13 points)

a) Let A be a $n \times n$, $n \geq 2$, symmetric, invertible matrix with real positive elements. Show that $z_n \leq n^2 - 2n$, where z_n is the number of zero elements in A^{-1}.

b) How many zero elements are there in the inverse of the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & 2 & \dots & 2 \\ 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 2 & \dots & 2 \\ \multicolumn{6}{c}{\dotfill} \\ 1 & 2 & 1 & 2 & \dots & \dots \end{pmatrix} ?$$

Solution. Denote by a_{ij} and b_{ij} the elements of A and A^{-1}, respectively. Then for $k \neq m$ we have $\sum\limits_{i=0}^{n} a_{ki}b_{im} = 0$ and from the positivity of a_{ij} we conclude that at least one of $\{b_{im} : i = 1, 2, \dots, n\}$ is positive and at least one is negative. Hence we have at least two non-zero elements in every column of A^{-1}. This proves part a). For part b) all b_{ij} are zero except $b_{1,1} = 2$, $b_{n,n} = (-1)^n$, $b_{i,i+1} = b_{i+1,i} = (-1)^i$ for $i = 1, 2, \dots, n-1$.

Problem 2. (13 points)

Let $f \in C^1(a, b)$, $\lim\limits_{x \to a+} f(x) = +\infty$, $\lim\limits_{x \to b-} f(x) = -\infty$ and $f'(x) + f^2(x) \geq -1$ for $x \in (a, b)$. Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

Solution. From the inequality we get

$$\frac{d}{dx}(\operatorname{arctg} f(x) + x) = \frac{f'(x)}{1 + f^2(x)} + 1 \geq 0$$

for $x \in (a, b)$. Thus $\operatorname{arctg} f(x) + x$ is non-decreasing in the interval and using the limits we get $\frac{\pi}{2} + a \leq -\frac{\pi}{2} + b$. Hence $b - a \geq \pi$. One has equality for $f(x) = \operatorname{cotg} x$, $a = 0$, $b = \pi$.

Problem 3. (13 points)

Given a set S of $2n-1$, $n \in \mathbb{N}$, different irrational numbers. Prove that there are n different elements $x_1, x_2, \dots, x_n \in S$ such that for all non-negative rational numbers $a_1, a_2, \dots, a_n$ with $a_1 + a_2 + \cdots + a_n > 0$ we have that $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ is an irrational number.

Solution. Let $\mathbb{I}$ be the set of irrational numbers, $\mathbb{Q}$ – the set of rational numbers, $\mathbb{Q}^+ = \mathbb{Q} \cap [0, \infty)$. We work by induction. For $n = 1$ the statement is trivial. Let it be true for $n-1$. We start to prove it for n. From the induction argument there are $n-1$ different elements $x_1, x_2, \dots, x_{n-1} \in S$ such that

$$\text{(1)} \qquad \begin{array}{l} a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1} \in \mathbb{I} \\ \text{for all } \ a_1, a_2, \dots, a_n \in \mathbb{Q}^+ \text{ with } a_1 + a_2 + \cdots + a_{n-1} > 0. \end{array}$$

Denote the other elements of S by $x_n, x_{n+1}, \dots, x_{2n-1}$. Assume the statement is not true for n. Then for $k = 0, 1, \dots, n-1$ there are $r_k \in \mathbb{Q}$ such that

$$\text{(2)} \qquad \sum_{i=1}^{n-1} b_{ik}x_i + c_kx_{n+k} = r_k \ \text{ for some } b_{ik}, c_k \in \mathbb{Q}^+, \ \sum_{i=1}^{n-1} b_{ik} + c_k > 0.$$

Also

$$\text{(3)} \qquad \sum_{k=0}^{n-1} d_kx_{n+k} = R \ \text{ for some } d_k \in \mathbb{Q}^+, \ \sum_{k=0}^{n-1} d_k > 0, \quad R \in \mathbb{Q}.$$

If in (2) $c_k = 0$ then (2) contradicts (1). Thus $c_k \neq 0$ and without loss of generality one may take $c_k = 1$. In (2) also $\sum\limits_{i=1}^{n-1} b_{ik} > 0$ in view of $x_{n+k} \in \mathbb{I}$. Replacing (2) in (3) we get

$$\sum_{k=0}^{n-1} d_k \left(-\sum_{i=1}^{n-1} b_{ik}x_i + r_k \right) = R \ \text{ or } \ \sum_{i=1}^{n-1} \left(\sum_{k=0}^{n-1} d_kb_{ik} \right) x_i \in \mathbb{Q},$$

which contradicts (1) because of the conditions on b's and d's.

Problem 4. (18 points)

Let $\alpha \in \mathbb{R} \setminus \{0\}$ and suppose that F and G are linear maps (operators) from $\mathbb{R}^n$ into $\mathbb{R}^n$ satisfying $F \circ G - G \circ F = \alpha F$.

a) Show that for all $k \in \mathbb{N}$ one has $F^k \circ G - G \circ F^k = \alpha k F^k$.

b) Show that there exists $k \geq 1$ such that $F^k = 0$.

Solution. For a) using the assumptions we have

$$\begin{aligned} F^k \circ G - G \circ F^k &= \sum_{i=1}^{k} \left(F^{k-i+1} \circ G \circ F^{i-1} - F^{k-i} \circ G \circ F^i \right) = \\ &= \sum_{i=1}^{k} F^{k-i} \circ (F \circ G - G \circ F) \circ F^{i-1} = \\ &= \sum_{i=1}^{k} F^{k-i} \circ \alpha F \circ F^{i-1} = \alpha k F^k. \end{aligned}$$

b) Consider the linear operator $L(F) = F \circ G - G \circ F$ acting over all $n \times n$ matrices F. It may have at most n^2 different eigenvalues. Assuming that $F^k \neq 0$ for every k we get that L has infinitely many different eigenvalues αk in view of a) – a contradiction.

Problem 5. (18 points)

a) Let $f \in C[0, b]$, $g \in C(\mathbb{R})$ and let g be periodic with period b. Prove that $\int_0^b f(x)g(nx)dx$ has a limit as $n \to \infty$ and

$$\lim_{n\to\infty} \int_0^b f(x)g(nx)dx = \frac{1}{b} \int_0^b f(x)dx \cdot \int_0^b g(x)dx.$$

b) Find

$$\lim_{n\to\infty} \int_0^{\pi} \frac{\sin x}{1 + 3\cos^2 nx} dx.$$

Solution. Set $\|g\|_1 = \int_0^b |g(x)|dx$ and

$$\omega(f, t) = \sup\{|f(x) - f(y)| \ : \ x, y \in [0, b], \ |x - y| \le t\}.$$

In view of the uniform continuity of f we have $\omega(f, t) \to 0$ as $t \to 0$. Using the periodicity of g we get

$$\begin{aligned} &\int_0^b f(x)g(nx)dx = \sum_{k=1}^{n} \int_{b(k-1)/n}^{bk/n} f(x)g(nx)dx \\ &= \sum_{k=1}^{n} f(bk/n) \int_{b(k-1)/n}^{bk/n} g(nx)dx + \sum_{k=1}^{n} \int_{b(k-1)/n}^{bk/n} \{f(x) - f(bk/n)\} g(nx)dx \\ &= \frac{1}{n} \sum_{k=1}^{n} f(bk/n) \int_0^b g(x)dx + O(\omega(f, b/n)\|g\|_1) \end{aligned}$$

$$= \frac{1}{b}\sum_{k=1}^{n}\int_{b(k-1)/n}^{bk/n} f(x)dx \int_0^b g(x)dx$$
$$+ \frac{1}{b}\sum_{k=1}^{n}\left(\frac{b}{n}f(bk/n) - \int_{b(k-1)/n}^{bk/n} f(x)dx\right)\int_0^b g(x)dx + O(\omega(f,b/n)\|g\|_1)$$
$$= \frac{1}{b}\int_0^b f(x)dx \int_0^b g(x)dx + O(\omega(f,b/n)\|g\|_1).$$

This proves a). For b) we set $b=\pi,\ \ f(x)=\sin x,\ \ g(x)=(1+3\cos^2 x)^{-1}$. From a) and

$$\int_0^\pi \sin x dx = 2, \qquad \int_0^\pi (1+3\cos^2 x)^{-1}dx = \frac{\pi}{2}$$

we get

$$\lim_{n\to\infty}\int_0^\pi \frac{\sin x}{1+3\cos^2 nx}dx = 1.$$

Problem 6. (25 points)

Let $f \in C^2[0,N]$ and $|f'(x)| < 1,\ f''(x) > 0$ for every $x \in [0,N]$. Let $0 \le m_0 < m_1 < \cdots < m_k \le N$ be integers such that $n_i = f(m_i)$ are also integers for $i = 0,1,\ldots,k$. Denote $b_i = n_i - n_{i-1}$ and $a_i = m_i - m_{i-1}$ for $i = 1,2,\ldots,k$.

a) Prove that

$$-1 < \frac{b_1}{a_1} < \frac{b_2}{a_2} < \cdots < \frac{b_k}{a_k} < 1.$$

b) Prove that for every choice of $A > 1$ there are no more than N/A indices j such that $a_j > A$.

c) Prove that $k \le 3N^{2/3}$ (i.e. there are no more than $3N^{2/3}$ integer points on the curve $y = f(x)$, $x \in [0,N]$).

Solution. a) For $i = 1,2,\ldots,k$ we have

$$b_i = f(m_i) - f(m_{i-1}) = (m_i - m_{i-1})f'(x_i)$$

for some $x_i \in (m_{i-1}, m_i)$. Hence $\dfrac{b_i}{a_i} = f'(x_i)$ and so $-1 < \dfrac{b_i}{a_i} < 1$. From the convexity of f we have that f' is increasing and $\dfrac{b_i}{a_i} = f'(x_i) < f'(x_{i+1}) = \dfrac{b_{i+1}}{a_{i+1}}$ because of $x_i < m_i < x_{i+1}$.

b) Set $S_A = \{j \in \{0, 1, \ldots, k\} \ : \ a_j > A\}$. Then

$$N \geq m_k - m_0 = \sum_{i=1}^{k} a_i \geq \sum_{j \in S_A} a_j > A|S_A|$$

and hence $|S_A| < N/A$.

c) All different fractions in $(-1, 1)$ with denominators less or equal A are no more $2A^2$. Using b) we get $k < N/A + 2A^2$. Put $A = N^{1/3}$ in the above estimate and get $k < 3N^{2/3}$.

Second day — July 30, 1994

Problem 1. (14 points)

Let $f \in C^1[a, b]$, $f(a) = 0$ and suppose that $\lambda \in \mathbb{R}$, $\lambda > 0$, is such that

$$|f'(x)| \leq \lambda |f(x)|$$

for all $x \in [a, b]$. Is it true that $f(x) = 0$ for all $x \in [a, b]$?

Solution. Assume that there is $y \in (a, b]$ such that $f(y) \neq 0$. Without loss of generality we have $f(y) > 0$. In view of the continuity of f there exists $c \in [a, y)$ such that $f(c) = 0$ and $f(x) > 0$ for $x \in (c, y]$. For $x \in (c, y]$ we have $|f'(x)| \leq \lambda f(x)$. This implies that the function $g(x) = \ln f(x) - \lambda x$ is not increasing in $(c, y]$ because of $g'(x) = \dfrac{f'(x)}{f(x)} - \lambda \leq 0$. Thus $\ln f(x) - \lambda x \geq \ln f(y) - \lambda y$ and $f(x) \geq e^{\lambda x - \lambda y} f(y)$ for $x \in (c, y]$. Thus

$$0 = f(c) = f(c + 0) \geq e^{\lambda c - \lambda y} f(y) > 0$$

— a contradiction. Hence one has $f(x) = 0$ for all $x \in [a, b]$.

Problem 2. (14 points)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = (x^2 - y^2) e^{-x^2 - y^2}$.

a) Prove that f attains its minimum and its maximum.

b) Determine all points (x, y) such that $\dfrac{\partial f}{\partial x}(x, y) = \dfrac{\partial f}{\partial y}(x, y) = 0$ and determine for which of them f has global or local minimum or maximum.

Solution. We have $f(1, 0) = e^{-1}$, $f(0, 1) = -e^{-1}$ and $te^{-t} \leq 2e^{-2}$ for $t \geq 2$. Therefore $|f(x, y)| \leq (x^2 + y^2) e^{-x^2 - y^2} \leq 2e^{-2} < e^{-1}$ for $(x, y) \notin M = \{(u, v) \ : \ u^2 + v^2 \leq 2\}$ and f cannot attain its minimum and its maximum outside M. Part a) follows from the compactness of M and the continuity of f. Let (x, y) be a point from part b). From $\frac{\partial f}{\partial x}(x, y) = 2x(1 - x^2 + y^2)e^{-x^2-y^2}$ we get

$$x(1 - x^2 + y^2) = 0. \tag{1}$$

Similarly

$$y(1 + x^2 - y^2) = 0. \tag{2}$$

All solutions (x, y) of the system (1), (2) are $(0, 0)$, $(0, 1)$, $(0, -1)$, $(1, 0)$ and $(-1, 0)$. One has $f(1, 0) = f(-1, 0) = e^{-1}$ and f has global maximum at the points $(1, 0)$ and $(-1, 0)$. One has $f(0, 1) = f(0, -1) = -e^{-1}$ and f has global minimum at the points $(0, 1)$ and $(0, -1)$. The point $(0, 0)$ is not an extrema point because of $f(x, 0) = x^2e^{-x^2} > 0$ if $x \neq 0$ and $f(y, 0) = -y^2e^{-y^2} < 0$ if $y \neq 0$.

Problem 3. (14 points)

Let f be a real-valued function with $n + 1$ derivatives at each point of $\mathbb{R}$. Show that for each pair of real numbers a, b, $a < b$, such that

$$\ln\left(\frac{f(b) + f'(b) + \cdots + f^{(n)}(b)}{f(a) + f'(a) + \cdots + f^{(n)}(a)}\right) = b - a$$

there is a number c in the open interval (a, b) for which

$$f^{(n+1)}(c) = f(c).$$

Note that ln denotes the natural logarithm.

Solution. Set $g(x) = \left(f(x) + f'(x) + \cdots + f^{(n)}(x)\right)e^{-x}$. From the assumption one get $g(a) = g(b)$. Then there exists $c \in (a, b)$ such that $g'(c) = 0$. Replacing in the last equality $g'(x) = \left(f^{(n+1)}(x) - f(x)\right)e^{-x}$ we finish the proof.

Problem 4. (18 points)

Let A be a $n \times n$ diagonal matrix with characteristic polynomial

$$(x - c_1)^{d_1}(x - c_2)^{d_2} \ldots (x - c_k)^{d_k},$$

where $c_1, c_2, \ldots, c_k$ are distinct (which means that c_1 appears d_1 times on the diagonal, c_2 appears d_2 times on the diagonal, etc. and $d_1 + d_2 + \cdots + d_k = n$).

Let V be the space of all $n \times n$ matrices B such that $AB = BA$. Prove that the dimension of V is

$$d_1^2 + d_2^2 + \cdots + d_k^2.$$

Solution. Set $A = (a_{ij})_{i,j=1}^n$, $B = (b_{ij})_{i,j=1}^n$, $AB = (x_{ij})_{i,j=1}^n$ and $BA = (y_{ij})_{i,j=1}^n$. Then $x_{ij} = a_{ii}b_{ij}$ and $y_{ij} = a_{jj}b_{ij}$. Thus $AB = BA$ is equivalent to $(a_{ii} - a_{jj})b_{ij} = 0$ for $i, j = 1, 2, \ldots, n$. Therefore $b_{ij} = 0$ if $a_{ii} \neq a_{jj}$ and b_{ij} may be arbitrary if $a_{ii} = a_{jj}$. The number of indices (i, j) for which $a_{ii} = a_{jj} = c_m$ for some $m = 1, 2, \ldots, k$ is d_m^2. This gives the desired result.

Problem 5. (18 points)

Let $x_1, x_2, \ldots, x_k$ be vectors of m-dimensional Euclidian space, such that $x_1 + x_2 + \cdots + x_k = 0$. Show that there exists a permutation π of the integers $\{1, 2, \ldots, k\}$ such that

$$\left\| \sum_{i=1}^{n} x_{\pi(i)} \right\| \leq \left(\sum_{i=1}^{k} \|x_i\|^2 \right)^{1/2} \quad \text{for each } n = 1, 2, \ldots, k.$$

Note that $\|\cdot\|$ denotes the Euclidian norm.

Solution. We define π inductively. Set $\pi(1) = 1$. Assume π is defined for $i = 1, 2, \ldots, n$ and also

$$\left\| \sum_{i=1}^{n} x_{\pi(i)} \right\|^2 \leq \sum_{i=1}^{n} \|x_{\pi(i)}\|^2. \tag{1}$$

Note (1) is true for $n = 1$. We choose $\pi(n+1)$ in a way that (1) is fulfilled with $n+1$ instead of n. Set $y = \sum\limits_{i=1}^{n} x_{\pi(i)}$ and $A = \{1, 2, \ldots, k\} \setminus \{\pi(i) : i = 1, 2, \ldots, n\}$. Assume that $(y, x_r) > 0$ for all $r \in A$. Then $\left(y, \sum\limits_{r \in A} x_r \right) > 0$ and in view of $y + \sum\limits_{r \in A} x_r = 0$ one gets $-(y, y) > 0$, which is impossible. Therefore there is $r \in A$ such that

$$(y, x_r) \leq 0. \tag{2}$$

Put $\pi(n+1) = r$. Then using (2) and (1) we have

$$\left\| \sum_{i=1}^{n+1} x_{\pi(i)} \right\|^2 = \|y + x_r\|^2 = \|y\|^2 + 2(y, x_r) + \|x_r\|^2 \leq \|y\|^2 + \|x_r\|^2 \leq$$

$$\leq \sum_{i=1}^{n} \|x_{\pi(i)}\|^2 + \|x_r\|^2 = \sum_{i=1}^{n+1} \|x_{\pi(i)}\|^2,$$

which verifies (1) for $n+1$. Thus we define π for every $n = 1, 2, \dots, k$. Finally from (1) we get

$$\left\| \sum_{i=1}^{n} x_{\pi(i)} \right\|^2 \leq \sum_{i=1}^{n} \|x_{\pi(i)}\|^2 \leq \sum_{i=1}^{k} \|x_i\|^2.$$

Problem 6. (22 points)

Find $\lim\limits_{N\to\infty} \dfrac{\ln^2 N}{N} \sum\limits_{k=2}^{N-2} \dfrac{1}{\ln k \cdot \ln(N-k)}$. Note that ln denotes the natural logarithm.

Solution. Obviously

$$(1) \qquad A_N = \frac{\ln^2 N}{N} \sum_{k=2}^{N-2} \frac{1}{\ln k \cdot \ln(N-k)} \geq \frac{\ln^2 N}{N} \cdot \frac{N-3}{\ln^2 N} = 1 - \frac{3}{N}.$$

Take M, $2 \leq M < N/2$. Then using that $\dfrac{1}{\ln k \cdot \ln(N-k)}$ is decreasing in $[2, N/2]$ and the symmetry with respect to $N/2$ one get

$$A_N = \frac{\ln^2 N}{N} \left\{ \sum_{k=2}^{M} + \sum_{k=M+1}^{N-M-1} + \sum_{k=N-M}^{N-2} \right\} \frac{1}{\ln k \cdot \ln(N-k)} \leq$$

$$\leq \frac{\ln^2 N}{N} \left\{ 2\frac{M-1}{\ln 2 \cdot \ln(N-2)} + \frac{N-2M-1}{\ln M \cdot \ln(N-M)} \right\} \leq$$

$$\leq \frac{2}{\ln 2} \cdot \frac{M \ln N}{N} + \left(1 - \frac{2M}{N}\right) \frac{\ln N}{\ln M} + O\left(\frac{1}{\ln N}\right).$$

Choose $M = \left[\dfrac{N}{\ln^2 N}\right] + 1$ to get

$$(2) \quad A_N \leq \left(1 - \frac{2}{N \ln^2 N}\right) \frac{\ln N}{\ln N - 2\ln\ln N} + O\left(\frac{1}{\ln N}\right) \leq 1 + O\left(\frac{\ln\ln N}{\ln N}\right).$$

Estimates (1) and (2) give

$$\lim_{N\to\infty} \frac{\ln^2 N}{N} \sum_{k=2}^{N-2} \frac{1}{\ln k \cdot \ln(N-k)} = 1.$$