

IMC2023: Problems on Day 1

Problem 1. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ that have a continuous second derivative and for which the equality $f(7x + 1) = 49f(x)$ holds for all $x \in \mathbb{R}$.

Alex Avdiushenko, Neapolis University Paphos, Cyprus

Problem 2. Let A , B and C be $n \times n$ matrices with complex entries satisfying

$$A^2 = B^2 = C^2 \quad \text{and} \quad B^3 = ABC + 2I.$$

Prove that $A^6 = I$.

Mike Daas, Universiteit Leiden

Problem 3. Find all polynomials P in two variables with real coefficients satisfying the identity

$$P(x, y)P(z, t) = P(xz - yt, xt + yz).$$

Giorgi Arabidze, Free University of Tbilisi, Georgia

Problem 4. Let p be a prime number and let k be a positive integer. Suppose that the numbers $a_i = i^k + i$ for $i = 0, 1, \dots, p-1$ form a complete residue system modulo p . What is the set of possible remainders of a_2 upon division by p ?

Tigran Hakobyan, Yerevan State University, Armenia

Problem 5. Fix positive integers n and k such that $2 \leq k \leq n$ and a set M consisting of n fruits. A *permutation* is a sequence $x = (x_1, x_2, \dots, x_n)$ such that $\{x_1, \dots, x_n\} = M$. Ivan *prefers* some (at least one) of these permutations. He realized that for every preferred permutation x , there exist k indices $i_1 < i_2 < \dots < i_k$ with the following property: for every $1 \leq j < k$, if he swaps x_{i_j} and $x_{i_{j+1}}$, he obtains another preferred permutation.

Prove that he prefers at least $k!$ permutations.

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