

IMC2021: Problems on Day 1

Problem 1. Let A be a real $n \times n$ matrix such that $A^3 = 0$.

(a) Prove that there is a unique real $n \times n$ matrix X that satisfies the equation

$$X + AX + XA^2 = A.$$

(b) Express X in terms of A .

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Problem 2. Let n and k be fixed positive integers, and let a be an arbitrary non-negative integer. Choose a random k -element subset X of $\{1, 2, \dots, k + a\}$ uniformly (i.e., all k -element subsets are chosen with the same probability) and, independently of X , choose a random n -element subset Y of $\{1, \dots, k + n + a\}$ uniformly.

Prove that the probability

$$P\left(\min(Y) > \max(X)\right)$$

does not depend on a .

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Problem 3. We say that a positive real number d is *good* if there exists an infinite sequence $a_1, a_2, a_3, \dots \in (0, d)$ such that for each n , the points $a_1, \dots, a_n$ partition the interval $[0, d]$ into segments of length at most $1/n$ each. Find

$$\sup \left\{ d \mid d \text{ is good} \right\}.$$

Josef Tkadlec

Problem 4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Suppose that for every $\varepsilon > 0$, there exists a function $g : \mathbb{R} \rightarrow (0, \infty)$ such that for every pair (x, y) of real numbers,

$$\text{if } |x - y| < \min \{g(x), g(y)\}, \quad \text{then } |f(x) - f(y)| < \varepsilon.$$

Prove that f is the pointwise limit of a sequence of continuous $\mathbb{R} \rightarrow \mathbb{R}$ functions, i.e., there is a sequence $h_1, h_2, \dots$ of continuous $\mathbb{R} \rightarrow \mathbb{R}$ functions such that $\lim_{n \rightarrow \infty} h_n(x) = f(x)$ for every $x \in \mathbb{R}$.

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