IMC2017: Problems on Day 2

6. Let $f : [0; +\infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x\to+\infty} f(x) = L$ exists (it may be finite or infinite). Prove that

$$\lim_{n\to\infty} \int_0^1 f(nx)\,dx = L.$$

Proposed by: Alexandr Bolbot, Novosibirsk State Uni

Hint Sol

7. Let $p(x)$ be a nonconstant polynomial with real coefficients. For every positive integer~n, let

$$q_n(x) = (x+1)^n p(x) + x^n p(x+1).$$

Prove that there are only finitely many numbers n such that all roots of $q_n(x)$ are real.

Proposed by: Alexandr Bolbot, Novosibirsk State Uni

Hint Sol

8. Define the sequence $A_1, A_2, \dots$ of matrices by the following recurrence:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & I_{2^n} \\ I_{2^n} & A_n \end{pmatrix} \quad (n = 1, 2, \dots)$$

where I_m is the $m \times m$ identity matrix.

Prove that A_n has $n+1$ distinct integer eigenvalues $\lambda_0 < \lambda_1 < \dots < \lambda_n$ with multiplicities $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$, respectively.

Proposed by: Snježana Majstorović, University of J. J. Strossmayer in

Sol

9. Define the sequence $f_1, f_2, \dots : [0,1) \to \mathbb{R}$ of continuously differentiable functions by the following recurrence:

$$f_1 = 1; \qquad f'_{n+1} = f_n f_{n+1} \quad \text{on } (0,1), \quad \text{and} \quad f_{n+1}(0) = 1.$$

Show that $\lim_{n\to\infty} f_n(x)$ exists for every $x \in [0,1)$ and determine the limit function.

Proposed by: Tomáš Bárta, Charles University,

Hint Sol

10. Let K be an equilateral triangle in the plane. Prove that for every $p > 0$ there exists an $\varepsilon > 0$ with the following property: If n is a positive integer, and $T_1, \dots,$ non-overlapping triangles inside K such that each of them is homothetic to K with a negative ratio, and

$$\sum_{\ell=1}^{n} \text{area}(T_\ell) > \text{area}(K) - \varepsilon,$$

then

$$\sum_{\ell=1}^{n} \text{perimeter}(T_\ell) > p.$$

Proposed by: Fedor Malyshev, Steklov Mathematical Institute and Ilya Bogdanov, Moscow Institute of Physics and Tech

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