IMC2017: Problems on Day 1

1. Determine all complex numbers λ for which there exist a positive integer n and a real $n \times n$ matrix A such that $A^2 = A^T$ and λ is an eigenvalue of A.

Proposed by: Alexandr Bolbot, Novosibirsk State Uni

Hint Sol

2. Let $f:\mathbb{R} \to (0,\infty)$ be a differentiable function, and suppose that there exists a constant $L > 0$ such that

$$|f'(x) - f'(y)| \le L|x - y|$$

for all x, y. Prove that

$$\left(f'(x)\right)^2 < 2Lf(x)$$

holds for all x.

Proposed by: Jan Šustek, University of C

Hint Sol

3. For any positive integer m, denote by $P(m)$ the product of positive divisors of m (e.g. $P(6) = 36$). For every positive integer n define the sequence

$$a_1(n) = n, \qquad a_{k+1}(n) = P(a_k(n)) \quad (k = 1, 2, \ldots, 2016).$$

Determine whether for every set $S \subseteq \{1, 2, \ldots, 2017\}$, there exists a positive integer n such that the following condition is satisfied:

For every k *with* $1 \le k \le 2017$, *the number* $a_k(n)$ *is a perfect square if and only if* $k \in S$.

Proposed by: Matko Ljulj, University of Z

Hint Sol

4. There are n people in a city, and each of them has exactly 1000 friends (friendship is always symmetric). Prove that it is possible to select a group S of people su at least $n/2017$ persons in S have exactly two friends in S.

Proposed by: Rooholah Majdodin and Fedor Petrov, St. Petersburg State Uni

Hint Sol

5. Let k and n be positive integers with $n \ge k^2 - 3k + 4$, and let

$$f(z) = z^{n-1} + c_{n-2}z^{n-2} + \ldots + c_0$$

be a polynomial with complex coefficients such that

$$c_0c_{n-2} = c_1c_{n-3} = \ldots = c_{n-2}c_0 = 0.$$

Prove that $f(z)$ and $z^n - 1$ have at most $n - k$ common roots.

Proposed by: Vsevolod Lev and Fedor Petrov, St. Petersburg State Uni

Hint Sol

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