International Mathematics Competition for University Students 2016

IMC2016: Problems on Day 2

6. Let $(x_1, x_2, \ldots)$ be a sequence of positive real numbers satisfying $\sum_{n=1}^{\infty} \frac{x_n}{2n-1} = 1$. Prove that

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} \leq 2.$$

Proposed by Gerhard J. Woeginger, The Neth

Hint Sol

7. Today, Ivan the Confessor prefers continuous functions $f : [0,1] \to \mathbb{R}$ satisfying $f(x) + f(y) \geq |x - y|$ for all pairs $x, y \in [0,1]$. Find the minimum of $\int_0^1 f$ over preferred functions.

Proposed by Fedor Petrov, St. Petersburg State Uni

Hint Sol

8. Let n be a positive integer, and denote by $\mathbb{Z}_n$ the ring of integers modulo n. Suppose that there exists a function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ satisfying the following three prop

(i) $f(x) \neq x$,

(ii) $f(f(x)) = x$,

(iii) $f(f(f(x + 1) + 1) + 1) = x$ for all $x \in \mathbb{Z}_n$.

Prove that $n \equiv 2 \pmod 4$.

Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Ge

Hint Sol

9. Let k be a positive integer. For each nonnegative integer n, let $f(n)$ be the number of solutions $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ of the inequality $|x_1| + \ldots + |x_k| \leq n$. Prove that f every $n \geq 1$, we have $f(n-1)f(n+1) \leq f(n)^2$.

Proposed by Esteban Arreaga, Renan Finder and José Madrid, IMPA, Rio de

Hint Sol

10. Let A be a $n \times n$ complex matrix whose eigenvalues have absolute value at most 1. Prove that

$$\|A^n\| \leq \frac{n}{\ln 2} \|A\|^{n-1}.$$

(Here $\|B\| = \sup_{\|x\| \leq 1} \|Bx\|$ for every $n \times n$ matrix B and $\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$ for every complex vector $x \in \mathbb{C}^n$.)

Proposed by Ian Morris and Fedor Petrov, St. Petersburg State Uni

Hint Sol

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