IMC2016: Problems on Day 1

1. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that f has infinitely many zeros, but there is no $x \in (a, b)$ with $f(x) = f'(x) = 0$.

(a) Prove that $f(a)f(b) = 0$.

(b) Give an example of such a function on $[0, 1]$.

Proposed by Alexandr Bolbot, Novosibirsk State Uni

Hint Sol

2. Let k and n be positive integers. A sequence $(A_1, \dots, A_k)$ of $n \times n$ real matrices is *preferred* by **Ivan the Confessor** if $A_i^2 \neq 0$ for $1 \le i \le k$, but $A_i A_j = 0$ for $1 \le i, j \le k$ with $i \neq j$. Show that $k \le n$ in all preferred sequences, and give an example of a preferred sequence with $k = n$ for each n.

Proposed by Fedor Petrov, St. Petersburg State Uni

Hint Sol

3. Let n be a positive integer. Also let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers such that $a_i + b_i > 0$ for $i = 1, 2, \dots, n$. Prove that

$$\sum_{i=1}^{n} \frac{a_i b_i - b_i^2}{a_i + b_i} \le \frac{\sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i - \left(\sum_{i=1}^{n} b_i\right)^2}{\sum_{i=1}^{n} (a_i + b_i)}.$$

Proposed by Daniel Strzelecki, Nicolaus Copernicus University in Toruń,

Hint Sol

4. Let $n \ge k$ be positive integers, and let $\square$ be a family of finite sets with the following properties:

(i) $\square$ contains at least $\binom{n}{k} + 1$ distinct sets containing exactly k elements;

(ii) for any two sets $A, B \in \square$, their union $A \cup B$ also belongs to $\square$.

Prove that $\square$ contains at least three sets with at least n elements.

Proposed by Fedor Petrov, St. Petersburg State Uni

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5. Let S_n denote the set of permutations of the sequence $(1, 2, \dots, n)$. For every permutation $\pi = (\pi_1, \dots, \pi_n) \in S_n$, let $\mathrm{inv}(\pi)$ be the number of pairs $1 \le i < j \le n$
$\pi_i > \pi_j$; i.e. the number of inversions in π. Denote by $f(n)$ the number of permutations $\pi \in S_n$ for which $\mathrm{inv}(\pi)$ is divisible by $n + 1$.

Prove that there exist infinitely many primes p such that $f(p-1) > \frac{(p-1)!}{p}$, and infinitely many primes p such that $f(p-1) < \frac{(p-1)!}{p}$.

Proposed by Fedor Petrov, St. Petersburg State Uni

Hint Sol

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