

IMC2011, Blagoevgrad, Bulgaria

Day 2, July 31, 2011

Problem 1. Let $(a_n)_{n=0}^{\infty}$ be a sequence with $\frac{1}{2} < a_n < 1$ for all $n \geq 0$. Define the sequence $(x_n)_{n=0}^{\infty}$ by

$$x_0 = a_0, \quad x_{n+1} = \frac{a_{n+1} + x_n}{1 + a_{n+1}x_n} \quad (n \geq 0).$$

What are the possible values of $\lim_{n \rightarrow \infty} x_n$? Can such a sequence diverge?

(10 points)

Problem 2. An alien race has three genders: male, female, and emale. A *married triple* consists of three persons, one from each gender, who all like each other. Any person is allowed to belong to at most one married triple. A special feature of this race is that feelings are always mutual — if x likes y , then y likes x .

The race is sending an expedition to colonize a planet. The expedition has n males, n females, and n emales. It is known that every expedition member likes at least k persons of each of the two other genders. The problem is to create as many married triples as possible to produce healthy offspring so the colony could grow and prosper.

- a) Show that if n is even and $k = \frac{n}{2}$, then it might be impossible to create even one married triple.
- b) Show that if $k \geq \frac{3n}{4}$, then it is always possible to create n disjoint married triples, thus marrying all of the expedition members.

(10 points)

Problem 3. Determine the value of

$$\sum_{n=1}^{\infty} \ln \left(1 + \frac{1}{n} \right) \cdot \ln \left(1 + \frac{1}{2n} \right) \cdot \ln \left(1 + \frac{1}{2n+1} \right).$$

(10 points)

Problem 4. Let $f(x)$ be a polynomial with real coefficients of degree n . Suppose that $\frac{f(k) - f(m)}{k - m}$ is an integer for all integers $0 \leq k < m \leq n$. Prove that $a - b$ divides $f(a) - f(b)$ for all pairs of distinct integers a and b .

(10 points)

Problem 5. Let $F = A_0A_1 \dots A_n$ be a convex polygon in the plane. Define for all $1 \leq k \leq n-1$ the operation f_k which replaces F with a new polygon

$$f_k(F) = A_0 \dots A_{k-1}A'_kA_{k+1} \dots A_n,$$

where A'_k is the point symmetric to A_k with respect to the perpendicular bisector of $A_{k-1}A_{k+1}$. Prove that $(f_1 \circ f_2 \circ \dots \circ f_{n-1})^n(F) = F$. We suppose that all operations are well-defined on the polygons, to which they are applied, i.e. results are convex polygons again. ($A_0, A_1, \dots, A_n$ are the vertices of F in consecutive order.)

(10 points)