IMC 2018, Blagoevgrad, Bulgaria

Day 2, July 25, 2018

Problem 6. Let k be a positive integer. Find the smallest positive integer n for which there exist k nonzero vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$ such that for every pair i, j of indices with $|i-j| > 1$ the vectors v_i and v_j are orthogonal.

(10 points)

Problem 7. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers such that $a_0 = 0$ and

$$a_{n+1}^3 = a_n^2 - 8 \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Prove that the following series is convergent:

$$\sum_{n=0}^{\infty} |a_{n+1} - a_n|.$$

(10 points)

Problem 8. Let $\Omega = \{(x, y, z) \in \mathbb{Z}^3 : y + 1 \geq x \geq y \geq z \geq 0\}$. A frog moves along the points of Ω by jumps of length 1. For every positive integer n, determine the number of paths the frog can take to reach (n, n, n) starting from $(0, 0, 0)$ in exactly $3n$ jumps.

(10 points)

Problem 9. Determine all pairs $P(x)$, $Q(x)$ of complex polynomials with leading coefficient 1 such that $P(x)$ divides $Q(x)^2 + 1$ and $Q(x)$ divides $P(x)^2 + 1$.

(10 points)

Problem 10. For $R > 1$ let $\mathcal{D}_R = \{(a, b) \in \mathbb{Z}^2 \colon 0 < a^2 + b^2 < R\}$. Compute

$$\lim_{R \to \infty} \sum_{(a,b) \in \mathcal{D}_R} \frac{(-1)^{a+b}}{a^2 + b^2}.$$

(10 points)