IMC 2016, Blagoevgrad, Bulgaria

Day 2, July 28, 2016

Problem 1. Let $(x_1, x_2, \ldots)$ be a sequence of positive real numbers satisfying $\displaystyle\sum_{n=1}^{\infty} \frac{x_n}{2n-1} = 1$. Prove that

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} \le 2.$$

(Proposed by Gerhard J. Woeginger, The Netherlands)

Solution. By interchanging the sums,

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} = \sum_{1 \le n \le k} \frac{x_n}{k^2} = \sum_{n=1}^{\infty} \left(x_n \sum_{k=n}^{\infty} \frac{1}{k^2} \right).$$

Then we use the upper bound

$$\sum_{k=n}^{\infty} \frac{1}{k^2} \le \sum_{k=n}^{\infty} \frac{1}{k^2 - \frac{1}{4}} = \sum_{k=n}^{\infty} \left(\frac{1}{k-\frac{1}{2}} - \frac{1}{k+\frac{1}{2}} \right) = \frac{1}{n-\frac{1}{2}}$$

and get

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} = \sum_{n=1}^{\infty} \left(x_n \sum_{k=n}^{\infty} \frac{1}{k^2} \right) < \sum_{n=1}^{\infty} \left(x_n \cdot \frac{1}{n-\frac{1}{2}} \right) = 2 \sum_{n=1}^{\infty} \frac{x_n}{2n-1} = 2.$$

Problem 2. Today, Ivan the Confessor prefers continuous functions $f : [0,1] \to \mathbb{R}$ satisfying $f(x) + f(y) \ge |x-y|$ for all pairs $x, y \in [0,1]$. Find the minimum of $\int_0^1 f$ over all preferred functions.

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution. The minimum of $\int_0^1 f$ is $\frac{1}{4}$.

Applying the condition with $0 \le x \le \frac{1}{2}$, $y = x + \frac{1}{2}$ we get

$$f(x) + f(x + \tfrac{1}{2}) \ge \tfrac{1}{2}.$$

By integrating,

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^{1/2} \big(f(x) + f(x+\tfrac{1}{2}) \big)\,\mathrm{d}x \ge \int_0^{1/2} \tfrac{1}{2}\,\mathrm{d}x = \tfrac{1}{4}.$$

On the other hand, the function $f(x) = \left| x - \frac{1}{2} \right|$ satisfies the conditions because

$$|x-y| = \left| \left(x - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - y\right) \right| \le \left| x - \tfrac{1}{2} \right| + \left| \tfrac{1}{2} - y \right| = f(x) + f(y),$$

and establishes

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^{1/2} \left(\tfrac{1}{2} - x\right)\mathrm{d}x + \int_{1/2}^1 \left(x - \tfrac{1}{2}\right)\mathrm{d}x = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}.$$

Problem 3. Let n be a positive integer, and denote by $\mathbb{Z}_n$ the ring of integers modulo n. Suppose that there exists a function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ satisfying the following three properties:

(i) $f(x) \neq x$,

(ii) $f(f(x)) = x$,

(iii) $f(f(f(x+1)+1)+1) = x$ for all $x \in \mathbb{Z}_n$.

Prove that $n \equiv 2 \pmod 4$.

(Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Germany)

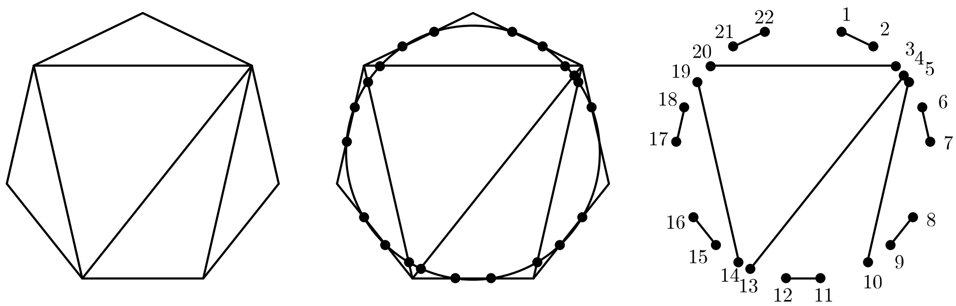
Solution. From property (ii) we can see that f is surjective, so f is a permutation of the elements in $\mathbb{Z}_n$, and its order is at most 2. Therefore, the permutation f is the product of disjoint transpositions of the form $\big(x, f(x)\big)$. Property (i) yields that this permutation has no fixed point, so n is be even, and the number of transpositions is precisely $n/2$.

Consider the permutation $g(x) = f(x+1)$. If g was odd then $g \circ g \circ g$ also would be odd. But property (iii) constraints that $g \circ g \circ g$ is the identity which is even. So g cannot be odd; g must be even. The cyclic permutation $h(x) = x - 1$ has order n, an even number, so h is odd. Then $f(x) = g \circ h$ is odd. Since f is the product of $n/2$ transpositions, this shows that $n/2$ must be odd, so $n \equiv 2 \pmod 4$.

Remark. There exists a function with properties (i–iii) for every $n \equiv 2 \pmod 4$. For $n = 2$ take $f(1) = 2$, $f(2) = 1$. Here we outline a possible construction for $n \geq 6$.

Let $n = 4k+2$, take a regular polygon with $k+2$ sides, and divide it into k triangles with $k-1$ diagonals. Draw a circle that intersects each side and each diagonal twice; altogether we have $4k+2$ intersections. Label the intersection points clockwise around the circle. On every side and diagonal we have two intersections; let f send them to each other.

This function f obviously satisfies properties (i) and (ii). For every x we either have $f(x+1) = x$ or the effect of adding 1 and taking f three times is going around the three sides of a triangle, so this function satisfies property (iii).



Problem 4. Let k be a positive integer. For each nonnegative integer n, let $f(n)$ be the number of solutions $(x_1, \dots, x_k) \in \mathbb{Z}^k$ of the inequality $|x_1| + \dots + |x_k| \leq n$. Prove that for every $n \geq 1$, we have $f(n-1)f(n+1) \leq f(n)^2$.

(Proposed by Esteban Arreaga, Renan Finder and José Madrid, IMPA, Rio de Janeiro)

Solution 1. We prove by induction on k. If $k = 1$ then we have $f(n) = 2n+1$ and the statement immediately follows from the AM-GM inequality.

Assume that $k \geq 2$ and the statement is true for $k-1$. Let $g(m)$ be the number of integer solutions of $|x_1| + \dots + |x_{k-1}| \leq m$; by the induction hypothesis $g(m-1)g(m+1) \leq g(m)^2$ holds; this can be transformed to

$$\frac{g(0)}{g(1)} \leq \frac{g(1)}{g(2)} \leq \frac{g(2)}{g(3)} \leq \dots.$$

For any integer constant c, the inequality $|x_1|+...+|x_{k-1}|+|c| \le n$ has $g\big(n-|c|\big)$ integer solutions. Therefore, we have the recurrence relation

$$f(n) = \sum_{c=-n}^{n} g\big(n-|c|\big) = g(n) + 2g(n-1) + ... + 2g(0).$$

It follows that

$$\frac{f(n-1)}{f(n)} = \frac{g(n-1)+2g(n-2)+...+2g(0)}{g(n)+2g(n-1)+...+2g(1)+2g(0)} \le$$

$$\le \frac{g(n)+g(n-1)+\big(g(n-1)+...+2g(0)+2\cdot 0\big)}{g(n+1)+g(n)+\big(g(n)+...+2g(1)+2g(0)\big)} = \frac{f(n)}{f(n+1)}$$

as required.

Solution 2. We first compute the generating function for $f(n)$:

$$\sum_{n=0}^{\infty} f(n)q^n = \sum_{(x_1,x_2,\dots,x_k)\in\mathbb{Z}^k} \sum_{c=0}^{\infty} q^{|x_1|+|x_2|+\dots+|x_k|+c} = \left(\sum_{x\in\mathbb{Z}} q^{|x|}\right)^k \frac{1}{1-q} = \frac{(1+q)^k}{(1-q)^{k+1}}.$$

For each $a=0,1,\dots$ denote by $g_a(n)$ $(n=0,1,2,\dots)$ the coefficients in the following expansion:

$$\frac{(1+q)^a}{(1-q)^{k+1}} = \sum_{n=0}^{\infty} g_a(n)q^n.$$

So it is clear that $g_{a+1}(n) = g_a(n)+g_a(n-1)$ $(n\ge 1)$, $g_a(0)=1$. Call a sequence of positive numbers $g(0)$, $g(1)$, $g(2)$, $\dots$ *good* if $\frac{g(n-1)}{g(n)}$ $(n=1,2,\dots)$ is an increasing sequence. It is straightforward to check that g_0 is good:

$$g_0(n) = \binom{k+n}{k}, \qquad \frac{g_0(n-1)}{g_0(n)} = \frac{n}{k+n}.$$

If g is a good sequence then a new sequence g' defined by $g'(0)=g(0)$, $g'(n)=g(n)+g(n-1)$ $(n\ge 1)$ is also good:

$$\frac{g'(n-1)}{g'(n)} = \frac{g(n-1)+g(n-2)}{g(n)+g(n-1)} = \frac{1+\frac{g(n-2)}{g(n-1)}}{1+\frac{g(n)}{g(n-1)}},$$

where define $g(-1)=0$. Thus we see that each of the sequences g_1, g_2, $\dots$, $g_k=f$ are good. So the desired inequality holds.

Problem 5. Let A be a $n\times n$ complex matrix whose eigenvalues have absolute value at most 1. Prove that

$$\|A^n\| \le \frac{n}{\ln 2}\|A\|^{n-1}.$$

(Here $\|B\| = \sup_{\|x\|\le 1}\|Bx\|$ for every $n\times n$ matrix B and $\|x\| = \sqrt{\sum_{i=1}^{n}|x_i|^2}$ for every complex vector $x\in\mathbb{C}^n$.)

(Proposed by Ian Morris and Fedor Petrov, St. Petersburg State University)

Solution 1. Let $r=\|A\|$. We have to prove $\|A^n\| \le \frac{n}{\ln 2}r^{n-1}$.

As is well-known, the matrix norm satisfies $\|XY\|\le\|X\|\cdot\|Y\|$ for any matrices X,Y, and as a simple consequence, $\|A^k\|\le\|A\|^k = r^k$ for every positive integer k.

Let $\chi(t) = (t-\lambda_1)(t-\lambda_2)\dots(t-\lambda_n) = t^n + c_1t^{n-1} + \dots + c_n$ be the characteristic polynomial of A. From Vieta's formulas we get

$$|c_k| = \left|\sum_{1\le i_1<\ldots<i_k\le n} \lambda_{i_1}\cdots\lambda_{i_k}\right| \le \sum_{1\le i_1<\ldots<i_k\le n} |\lambda_{i_1}\cdots\lambda_{i_k}| \le \binom{n}{k} \quad (k=1,2,\dots,n)$$

By the Cayley–Hamilton theorem we have $\chi(A) = 0$, so

$$\|A^n\| = \|c_1 A^{n-1} + \cdots + c_n\| \le \sum_{k=1}^{n} \binom{n}{k} \|A^k\| \le \sum_{k=1}^{n} \binom{n}{k} r^k = (1+r)^n - r^n.$$

Combining this with the trivial estimate $\|A^n\| \le r^n$, we have

$$\|A^n\| \le \min\big(r^n, (1+r)^n - r^n\big).$$

Let $r_0 = \frac{1}{\sqrt[n]{2}-1}$; it is easy to check that the two bounds are equal if $r = r_0$, moreover

$$r_0 = \frac{1}{e^{\ln 2/n} - 1} < \frac{n}{\ln 2}.$$

For $r \le r_0$ apply the trivial bound:

$$\|A^n\| \le r^n \le r_0 \cdot r^{n-1} < \frac{n}{\ln 2} r^{n-1}.$$

For $r > r_0$ we have

$$\|A^n\| \le (1+r)^n - r^n = r^{n-1} \cdot \frac{(1+r)^n - r^n}{r^{n-1}}.$$

Notice that the function $f(r) = \frac{(1+r)^n - r^n}{r^{n-1}}$ is decreasing because the numerator has degree $n-1$ and all coefficients are positive, so

$$\frac{(1+r)^n - r^n}{r^{n-1}} < \frac{(1+r_0)^n - r_0^n}{r_0^{n-1}} = r_0\big((1+1/r_0)^n - 1\big) = r_0 < \frac{n}{\ln 2},$$

so $\|A^n\| < \frac{n}{\ln 2} r^{n-1}$.

Solution 2. We will use the following facts which are easy to prove:

- For any square matrix A there exists a unitary matrix U such that UAU^{-1} is upper-triangular.
- For any matrices A, B we have $\|A\| \le \|(A|B)\|$ and $\|B\| \le \|(A|B)\|$ where $(A|B)$ is the matrix whose columns are the columns of A and the columns of B.
- For any matrices A, B we have $\|A\| \le \|\left(\frac{A}{B}\right)\|$ and $\|B\| \le \|\left(\frac{A}{B}\right)\|$ where $\left(\frac{A}{B}\right)$ is the matrix whose rows are the rows of A and the rows of B.
- Adding a zero row or a zero column to a matrix does not change its norm.

We will prove a stronger inequality

$$\|A^n\| \le n\|A\|^{n-1}$$

for any $n \times n$ matrix A whose eigenvalues have absolute value at most 1. We proceed by induction on n. The case $n = 1$ is trivial. Without loss of generality we can assume that the matrix A is upper-triangular. So we have

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

Note that the eigenvalues of A are precisely the diagonal entries. We split A as the sum of 3 matrices, $A = X + Y + Z$ as follows:

$$X = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

Denote by A' the matrix obtained from A by removing the first row and the first column:

$$A' = \begin{pmatrix} a_{22} & \cdots & a_{2n} \\ \cdots & & \cdots \\ 0 & \cdots & a_{nn} \end{pmatrix}.$$

We have $\|X\| \leq 1$ because $|a_{11}| \leq 1$. We also have

$$\|A'\| = \|Z\| \leq \|Y + Z\| \leq \|A\|.$$

Now we decompose A^n as follows:

$$A^n = XA^{n-1} + (Y + Z)A^{n-1}.$$

We substitute $A = X + Y + Z$ in the second term and expand the parentheses. Because of the following identities:

$$Y^2 = 0, \quad YX = 0, \quad ZY = 0, \quad ZX = 0$$

only the terms YZ^{n-1} and Z^n survive. So we have

$$A^n = XA^{n-1} + (Y + Z)Z^{n-1}.$$

By the induction hypothesis we have $\|A'^{n-1}\| \leq (n-1)\|A'\|^{n-2}$, hence $\|Z^{n-1}\| \leq (n-1)\|Z\|^{n-2} \leq (n-1)\|A\|^{n-2}$. Therefore

$$\|A^n\| \leq \|XA^{n-1}\| + \|(Y + Z)Z^{n-1}\| \leq \|A\|^{n-1} + (n-1)\|Y + Z\|\|A\|^{n-2} \leq n\|A\|^{n-1}.$$