IMC 2016, Blagoevgrad, Bulgaria

Day 2, July 28, 2016

Problem 6. Let $(x_1, x_2, \ldots)$ be a sequence of positive real numbers satisfying $\displaystyle\sum_{n=1}^{\infty} \frac{x_n}{2n-1} = 1$. Prove that

$$\sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{x_n}{k^2} \leq 2.$$

(10 points)

Problem 7. Today, Ivan the Confessor prefers continuous functions $f : [0,1] \to \mathbb{R}$ satisfying $f(x) + f(y) \geqslant |x - y|$ for all pairs $x, y \in [0,1]$. Find the minimum of $\int_0^1 f$ over all preferred functions.

(10 points)

Problem 8. Let n be a positive integer, and denote by $\mathbb{Z}_n$ the ring of integers modulo n. Suppose that there exists a function $f : \mathbb{Z}_n \to \mathbb{Z}_n$ satisfying the following three properties:

(i) $f(x) \neq x$,

(ii) $f(f(x)) = x$,

(iii) $f(f(f(x+1)+1)+1) = x$ for all $x \in \mathbb{Z}_n$.

Prove that $n \equiv 2 \pmod 4$.

(10 points)

Problem 9. Let k be a positive integer. For each nonnegative integer n, let $f(n)$ be the number of solutions $(x_1, \ldots, x_k) \in \mathbb{Z}^k$ of the inequality $|x_1| + \ldots + |x_k| \leq n$. Prove that for every $n \geq 1$, we have $f(n-1)f(n+1) \leq f(n)^2$.

(10 points)

Problem 10. Let A be a $n \times n$ complex matrix whose eigenvalues have absolute value at most 1. Prove that

$$\|A^n\| \leq \frac{n}{\ln 2} \|A\|^{n-1}.$$

(Here $\|B\| = \sup_{\|x\| \leq 1} \|Bx\|$ for every $n \times n$ matrix B and $\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$ for every complex vector $x \in \mathbb{C}^n$.)

(10 points)