IMC 2015, Blagoevgrad, Bulgaria

Day 1, July 29, 2015

Problem 1. For any integer $n \geq 2$ and two $n \times n$ matrices with real entries A, B that satisfy the equation

$$A^{-1} + B^{-1} = (A+B)^{-1}$$

prove that $\det(A) = \det(B)$.

Does the same conclusion follow for matrices with complex entries?

(Proposed by Zbigniew Skoczylas, Wrocław University of Technology)

Solution. Multiplying the equation by $(A+B)$ we get

$$I = (A+B)(A+B)^{-1} = (A+B)(A^{-1}+B^{-1}) =$$
$$= AA^{-1} + AB^{-1} + BA^{-1} + BB^{-1} = I + AB^{-1} + BA^{-1} + I$$
$$AB^{-1} + BA^{-1} + I = 0.$$

Let $X = AB^{-1}$; then $A = XB$ and $BA^{-1} = X^{-1}$, so we have $X + X^{-1} + I = 0$; multiplying by $(X-I)X$,

$$0 = (X-I)X \cdot (X + X^{-1} + I) = (X-I) \cdot (X^2 + X + I) = X^3 - I.$$

Hence,

$$X^3 = I$$
$$(\det X)^3 = \det(X^3) = \det I = 1$$
$$\det X = 1$$
$$\det A = \det(XB) = \det X \cdot \det B = \det B.$$

In case of complex matrices the statement is false. Let $\omega = \frac{1}{2}(-1 + i\sqrt{3})$. Obviously $\omega \notin \mathbb{R}$ and $\omega^3 = 1$, so $0 = 1 + \omega + \omega^2 = 1 + \omega + \overline{\omega}$.

Let $A = I$ and let B be a diagonal matrix with all entries along the diagonal equal to either ω or $\overline{\omega} = \omega^2$ such a way that $\det(B) \neq 1$ (if n is not divisible by 3 then one may set $B = \omega I$). Then $A^{-1} = I$, $B^{-1} = \overline{B}$. Obviously $I + B + \overline{B} = 0$ and

$$(A+B)^{-1} = (-\overline{B})^{-1} = -B = I + \overline{B} = A^{-1} + B^{-1}.$$

By the choice of A and B, $\det A = 1 \neq \det B$.

Problem 2. For a positive integer n, let $f(n)$ be the number obtained by writing n in binary and replacing every 0 with 1 and vice versa. For example, $n = 23$ is 10111 in binary, so $f(n)$ is 1000 in binary, therefore $f(23) = 8$. Prove that

$$\sum_{k=1}^{n} f(k) \le \frac{n^2}{4}.$$

When does equality hold?

(Proposed by Stephan Wagner, Stellenbosch University)

Solution. If r and k are positive integers with $2^{r-1} \le k < 2^r$ then k has r binary digits, so $k + f(k) = \overline{\underbrace{11\dots1}_{r}}^{(2)} = 2^r - 1$.

Assume that $2^{s-1} - 1 \le n \le 2^s - 1$. Then

$$\frac{n(n+1)}{2} + \sum_{k=1}^{n} f(k) = \sum_{k=1}^{n} (k + f(k)) =$$

$$= \sum_{r=1}^{s-1} \sum_{2^{r-1} \le k < 2^r} (k + f(k)) + \sum_{2^{s-1} \le k \le n} (k + f(k)) =$$

$$= \sum_{r=1}^{s-1} 2^{r-1} \cdot (2^r - 1) + (n - 2^{s-1} + 1) \cdot (2^s - 1) =$$

$$= \sum_{r=1}^{s-1} 2^{2r-1} - \sum_{r=1}^{s-1} 2^{r-1} + (n - 2^{s-1} + 1)(2^s - 1) =$$

$$= \tfrac{2}{3}(4^{s-1} - 1) - (2^{s-1} - 1) + (2^s - 1)n - 2^{2s-1} + 3 \cdot 2^{s-1} - 1 =$$

$$= (2^s - 1)n - \tfrac{1}{3}4^s + 2^s - \tfrac{2}{3}$$

and therefore

$$\frac{n^2}{4} - \sum_{k=1}^{n} f(k) = \frac{n^2}{4} - \left((2^s - 1)n - \tfrac{1}{3}4^s + 2^s - \tfrac{2}{3} - \frac{n(n+1)}{2} \right) =$$

$$= \tfrac{3}{4}n^2 - (2^s - \tfrac{3}{2})n + \tfrac{1}{3}4^s - 2^s + \tfrac{2}{3} =$$

$$= \frac{3}{4} \left(n - \frac{2^{s+1} - 2}{3} \right) \left(n - \frac{2^{s+1} - 4}{3} \right).$$

Notice that the difference of the last two factors is less than 1, and one of them must be an integer: $\frac{2^{s+1}-2}{3}$ is integer if s is even, and $\frac{2^{s+1}-4}{3}$ is integer if s is odd. Therefore, either one of them is 0, resulting a zero product, or both factors have the same sign, so the product is strictly positive. This solves the problem and shows that equality occurs if $n = \dfrac{2^{s+1} - 2}{3}$ (s is even) or $n = \dfrac{2^{s+1} - 4}{3}$ (s is odd).

Problem 3. Let $F(0)=0$, $F(1)=\frac{3}{2}$, and $F(n)=\frac{5}{2}F(n-1)-F(n-2)$ for $n\ge 2$.

Determine whether or not $\displaystyle\sum_{n=0}^{\infty}\frac{1}{F(2^n)}$ is a rational number.

(Proposed by Gerhard Woeginger, Eindhoven University of Technology)

Solution 1. The characteristic equation of our linear recurrence is $x^2-\frac{5}{2}x+1=0$, with roots $x_1=2$ and $x_2=\frac{1}{2}$. So $F(n)=a\cdot 2^n+b\cdot(\frac{1}{2})^n$ with some constants a,b. By $F(0)=0$ and $F(1)=\frac{3}{2}$, these constants satisfy $a+b=0$ and $2a+\frac{b}{2}=\frac{3}{2}$. So $a=1$ and $b=-1$, and therefore

$$F(n)=2^n-2^{-n}.$$

Observe that

$$\frac{1}{F(2^n)}=\frac{2^{2^n}}{(2^{2^n})^2-1}=\frac{1}{2^{2^n}-1}-\frac{1}{(2^{2^n})^2-1}=\frac{1}{2^{2^n}-1}-\frac{1}{2^{2^{n+1}}-1},$$

so

$$\sum_{n=0}^{\infty}\frac{1}{F(2^n)}=\sum_{n=0}^{\infty}\left(\frac{1}{2^{2^n}-1}-\frac{1}{2^{2^{n+1}}-1}\right)=\frac{1}{2^{2^0}-1}=1.$$

Hence the sum takes the value 1, which is rational.

Solution 2. As in the first solution we find that $F(n)=2^n-2^{-n}$. Then

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{F(2^n)} &= \sum_{n=0}^{\infty}\frac{1}{2^{2^n}-2^{-2^n}} = \sum_{n=0}^{\infty}\frac{(\frac{1}{2})^{2^n}}{1-(\frac{1}{2})^{2^{n+1}}}\\ &= \sum_{n=0}^{\infty}(\tfrac{1}{2})^{2^n}\sum_{k=0}^{\infty}\left((\tfrac{1}{2})^{2^{n+1}}\right)^k = \sum_{n=0}^{\infty}(\tfrac{1}{2})^{2^n}\sum_{k=0}^{\infty}(\tfrac{1}{2})^{2k\cdot 2^n}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(\tfrac{1}{2})^{2^n(2k+1)} = \sum_{m=1}^{\infty}(\tfrac{1}{2})^m = 1.\end{aligned}$$

(Here we used the fact that every positive integer m has a unique representation $m=2^n(2k+1)$ with non-negative integers n and k.)

This shows that the series converges to 1.

Problem 4. Determine whether or not there exist 15 integers $m_1,\dots,m_{15}$ such that

$$\sum_{k=1}^{15} m_k\cdot\arctan(k)=\arctan(16). \qquad (1)$$

(Proposed by Gerhard Woeginger, Eindhoven University of Technology)

Solution. We show that such integers $m_1,\dots,m_{15}$ do not exist.

Suppose that (1) is satisfied by some integers $m_1,\dots,m_{15}$. Then the argument of the complex number $z_1=1+16i$ coincides with the argument of the complex number

$$z_2 \;=\; (1+i)^{m_1}\,(1+2i)^{m_2}\,(1+3i)^{m_3}\;\cdots\cdots\;(1+15i)^{m_{15}}.$$

Therefore the ratio $R=z_2/z_1$ is real (and not zero). As $\operatorname{Re} z_1=1$ and $\operatorname{Re} z_2$ is an integer, R is a nonzero integer.

By considering the squares of the absolute values of z_1 and z_2, we get

$$(1+16^2)R^2 = \prod_{k=1}^{15}(1+k^2)^{m_k}.$$

Notice that $p = 1 + 16^2 = 257$ is a prime (the fourth Fermat prime), which yields an easy contradiction through p-adic valuations: all prime factors in the right hand side are strictly below p (as $k < 16$ implies $1 + k^2 < p$). On the other hand, in the left hand side the prime p occurs with an odd exponent.

Problem 5. Let $n \geq 2$, let $A_1, A_2, \dots, A_{n+1}$ be $n+1$ points in the n-dimensional Euclidean space, not lying on the same hyperplane, and let B be a point strictly inside the convex hull of $A_1, A_2, \dots, A_{n+1}$. Prove that $\angle A_i B A_j > 90^\circ$ holds for at least n pairs (i, j) with $1 \leq i < j \leq n+1$.

(Proposed by Géza Kós, Eötvös University, Budapest)

Solution. Let $\mathbf{v}_i = \overrightarrow{BA_i}$. The condition $\angle A_i B A_j > 90^\circ$ is equivalent with $\mathbf{v}_i \cdot \mathbf{v}_j < 0$. Since B is an interior point of the simplex, there are some weights $w_1, \dots, w_{n+1} > 0$ with $\sum\limits_{i=1}^{n+1} w_i \mathbf{v}_i = \mathbf{0}$.

Let us build a graph on the vertices $1, \dots, n+1$. Let the vertices i and j be connected by an edge if $\mathbf{v}_i \cdot \mathbf{v}_j < 0$. We show that this graph is connected. Since every connected graph on $n+1$ vertices has at least n edges, this will prove the problem statement.

Suppose the contrary that the graph is not connected; then the vertices can be split in two disjoint nonempty sets, say V and W such that $V \cup W = \{1, 2, \dots, n+1\}$. Since there is no edge between the two vertex sets, we have $\mathbf{v}_i \cdot \mathbf{v}_j \geq 0$ for all $i \in V$ and $j \in W$.

Consider

$$0 = \left(\sum_{i \in V \cup W} w_i \mathbf{v}_i\right)^2 = \left(\sum_{i \in V} w_i \mathbf{v}_i\right)^2 + \left(\sum_{i \in W} w_i \mathbf{v}_i\right)^2 + 2\sum_{i \in V}\sum_{i \in W} w_i w_j (\mathbf{v}_i \cdot \mathbf{v}_j).$$

Notice that all terms are nonnegative on the right-hand side. Moreover, $\sum\limits_{i \in V} w_i \mathbf{v}_i \neq \mathbf{0}$ and $\sum\limits_{i \in W} w_i \mathbf{v}_i \neq \mathbf{0}$, so there are at least two strictly nonzero terms, contradiction.

Remark 1. The number n in the statement is sharp; if $\mathbf{v}_{n+1} = (1, 1, \dots, 1)$ and $v_i = (\underbrace{0, \dots, 0}_{i-1}, -1, \underbrace{0, \dots, 0}_{n-i})$ for $i = 1, \dots, n$ then $\mathbf{v}_i \cdot \mathbf{v}_j < 0$ holds only when $i = n+1$ or $j = n+1$.

Remark 2. The origin of the problem is here: `http://math.stackexchange.com/questions/476640/n-simplex-in-an-intersection-of-n-balls/789390`