IMC 2014, Blagoevgrad, Bulgaria

Day 1, July 31, 2014

Problem 1. Determine all pairs (a, b) of real numbers for which there exists a unique symmetric 2×2 matrix M with real entries satisfying $\operatorname{trace}(M) = a$ and $\det(M) = b$.

(Proposed by Stephan Wagner, Stellenbosch University)

Solution 1. Let the matrix be

$$M = \begin{bmatrix} x & z \\ z & y \end{bmatrix}.$$

The two conditions give us $x + y = a$ and $xy - z^2 = b$. Since this is symmetric in x and y, the matrix can only be unique if $x = y$. Hence $2x = a$ and $x^2 - z^2 = b$. Moreover, if (x, y, z) solves the system of equations, so does $(x, y, -z)$. So M can only be unique if $z = 0$. This means that $2x = a$ and $x^2 = b$, so $a^2 = 4b$.

If this is the case, then M is indeed unique: if $x + y = a$ and $xy - z^2 = b$, then

$$(x - y)^2 + 4z^2 = (x + y)^2 + 4z^2 - 4xy = a^2 - 4b = 0,$$

so we must have $x = y$ and $z = 0$, meaning that

$$M = \begin{bmatrix} a/2 & 0 \\ 0 & a/2 \end{bmatrix}$$

is the only solution.

Solution 2. Note that $\operatorname{trace}(M) = a$ and $\det(M) = b$ if and only if the two eigenvalues λ_1 and λ_2 of M are solutions of $x^2 - ax + b = 0$. If $\lambda_1 \neq \lambda_2$, then

$$M_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \text{and} \qquad M_2 = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix}$$

are two distinct solutions, contradicting uniqueness. Thus $\lambda_1 = \lambda_2 = \lambda = a/2$, which implies $a^2 = 4b$ once again. In this case, we use the fact that M has to be diagonalisable as it is assumed to be symmetric. Thus there exists a matrix T such that

$$M = T^{-1} \cdot \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \cdot T,$$

however this reduces to $M = \lambda(T^{-1} \cdot I \cdot T) = \lambda I$, which shows again that M is unique.

Problem 2. Consider the following sequence

$$(a_n)_{n=1}^{\infty} = (1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, \dots).$$

Find all pairs (α, β) of positive real numbers such that $\lim\limits_{n\to\infty} \dfrac{\sum\limits_{k=1}^{n} a_k}{n^\alpha} = \beta$.

(Proposed by Tomas Barta, Charles University, Prague)

Solution. Let $N_n = \binom{n+1}{2}$ (then a_{N_n} is the first appearance of number n in the sequence) and consider limit of the subsequence

$$b_{N_n} := \frac{\sum_{k=1}^{N_n} a_k}{N_n^\alpha} = \frac{\sum_{k=1}^{n} 1 + \cdots + k}{\binom{n+1}{2}^\alpha} = \frac{\sum_{k=1}^{n} \binom{k+1}{2}}{\binom{n+1}{2}^\alpha} = \frac{\binom{n+2}{3}}{\binom{n+1}{2}^\alpha} = \frac{\frac{1}{6}n^3(1+2/n)(1+1/n)}{(1/2)^\alpha n^{2\alpha}(1+1/n)^\alpha}.$$

We can see that $\lim\limits_{n\to\infty} b_{N_n}$ is positive and finite if and only if $\alpha = 3/2$. In this case the limit is equal to $\beta = \frac{\sqrt{2}}{3}$. So, this pair $(\alpha, \beta) = (\frac{3}{2}, \frac{\sqrt{2}}{3})$ is the only candidate for solution. We will show convergence of the original sequence for these values of α and β.

Let N be a positive integer in $[N_n+1, N_{n+1}]$, i.e., $N = N_n + m$ for some $1 \le m \le n+1$. Then we have

$$b_N = \frac{\binom{n+2}{3} + \binom{m+1}{2}}{\left(\binom{n+1}{2} + m\right)^{3/2}}$$

which can be estimated by

$$\frac{\binom{n+2}{3}}{\left(\binom{n+1}{2} + n\right)^{3/2}} \le b_N \le \frac{\binom{n+2}{3} + \binom{n+1}{2}}{\binom{n+1}{2}^{3/2}}.$$

Since both bounds converge to $\frac{\sqrt{2}}{3}$, the sequence b_N has the same limit and we are done.

Problem 3. Let n be a positive integer. Show that there are positive real numbers $a_0, a_1, \dots, a_n$ such that for each choice of signs the polynomial

$$\pm a_n x^n \pm a_{n-1}x^{n-1} \pm \cdots \pm a_1 x \pm a_0$$

has n distinct real roots.

(Proposed by Stephan Neupert, TUM, München)

Solution. We proceed by induction on n. The statement is trivial for $n = 1$. Thus assume that we have some $a_n, \dots, a_0$ which satisfy the conditions for some n. Consider now the polynomials

$$\tilde{P}(x) = \pm a_n x^{n+1} \pm a_{n-1}x^n \pm \ldots \pm a_1 x^2 \pm a_0 x$$

By induction hypothesis and $a_0 \neq 0$, each of these polynomials has $n+1$ distinct zeros, including the n nonzero roots of $\pm a_n x^n \pm a_{n-1}x^{n-1} \pm \ldots \pm a_1 x \pm a_0$ and 0. In particular none of the polynomials has a root which is a local extremum. Hence we can choose some $\varepsilon > 0$ such that for each such polynomial $\tilde{P}(x)$ and each of its local extrema s we have $|\tilde{P}(s)| > \varepsilon$. We claim that then each of the polynomials

$$P(x) = \pm a_n x^{n+1} \pm a_{n-1}x^n \pm \ldots \pm a_1 x^2 \pm a_0 x \pm \varepsilon$$

has exactly $n+1$ distinct zeros as well. As $\tilde{P}(x)$ has $n+1$ distinct zeros, it admits a local extremum at n points. Call these local extrema $-\infty = s_0 < s_1 < s_2 < \ldots < s_n < s_{n+1} = \infty$. Then for each $i \in \{0, 1, \ldots, n\}$ the values $\tilde{P}(s_i)$ and $\tilde{P}(s_{i+1})$ have opposite signs (with the obvious convention at infinity). By choice of ε the same holds true for $P(s_i)$ and $P(s_{i+1})$. Hence there is at least one real zero of $P(x)$ in each interval (s_i, s_{i+1}), i.e. $P(x)$ has at least (and therefore exactly) $n+1$ zeros. This shows that we have found a set of positive reals $a'_{n+1} = a_n, a'_n = a_{n-1}, \ldots, a'_1 = a_0, a'_0 = \varepsilon$ with the desired properties.

Problem 4. Let $n > 6$ be a perfect number, and let $n = p_1^{e_1} \cdots p_k^{e_k}$ be its prime factorisation with $1 < p_1 < \ldots < p_k$. Prove that e_1 is an even number.

A number n is *perfect* if $s(n) = 2n$, where $s(n)$ is the sum of the divisors of n.

(Proposed by Javier Rodrigo, Universidad Pontificia Comillas)

Solution. Suppose that e_1 is odd, contrary to the statement.

We know that $s(n) = \prod_{i=1}^{k}(1 + p_i + p_i^2 + \cdots + p_i^{e_i}) = 2n = 2p_1^{e_1} \cdots p_k^{e_k}$. Since e_1 is an odd number, $p_1 + 1$ divides the first factor $1 + p_1 + p_1^2 + \cdots + p_1^{e_1}$, so $p_1 + 1$ divides $2n$. Due to $p_1 + 1 > 2$, at least one of the primes $p_1, \ldots, p_k$ divides $p_1 + 1$. The primes $p_3, \ldots, p_k$ are greater than $p_1 + 1$ and p_1 cannot divide $p_1 + 1$, so p_2 must divide $p_1 + 1$. Since $p_1 + 1 < 2p_2$, this possible only if $p_2 = p_1 + 1$, therefore $p_1 = 2$ and $p_2 = 3$. Hence, $6|n$.

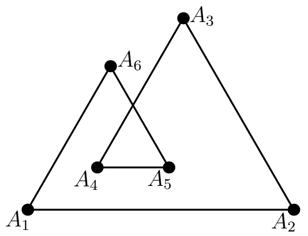
Now $n, \frac{n}{2}, \frac{n}{3}, \frac{n}{6}$ and 1 are distinct divisors of n, so

$$s(n) \geq n + \frac{n}{2} + \frac{n}{3} + \frac{n}{6} + 1 = 2n + 1 > 2n,$$

contradiction.

Remark. It is well-known that all even perfect numbers have the form $n = 2^{p-1}(2^p - 1)$ such that p and $2^p - 1$ are primes. So if e_1 is odd then $k = 2$, $p_1 = 2$, $p_2 = 2^p - 1$, $e_1 = p - 1$ and $e_2 = 1$. If $n > 6$ then $p > 2$ so p is odd and $e_1 = p - 1$ should be even.

Problem 5. Let $A_1A_2 \ldots A_{3n}$ be a closed broken line consisting of $3n$ line segments in the Euclidean plane. Suppose that no three of its vertices are collinear, and for each index $i = 1, 2, \ldots, 3n$, the triangle $A_iA_{i+1}A_{i+2}$ has counterclockwise orientation and $\angle A_iA_{i+1}A_{i+2} = 60°$, using the notation $A_{3n+1} = A_1$ and $A_{3n+2} = A_2$. Prove that the number of self-intersections of the broken line is at most $\frac{3}{2}n^2 - 2n + 1$.

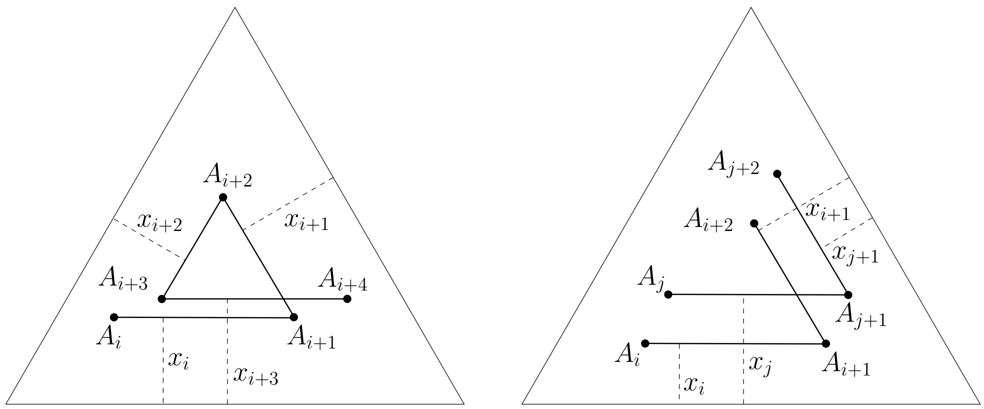


(Proposed by Martin Langer)

Solution. Place the broken line inside an equilateral triangle T such that their sides are parallel to the segments of the broken line. For every $i = 1, 2, \ldots, 3n$, denote by x_i the distance between the segment A_iA_{i+1} and that side of T which is parallel to A_iA_{i+1}. We will use indices modulo $3n$ everywhere.

It is easy to see that if $i \equiv j \pmod 3$ then the polylines $A_iA_{i+1}A_{i+2}$ and $A_jA_{j+1}A_{j+2}$ intersect at most once, and this is possible only if either $x_i < x_{i+1}$ and $x_j > x_{j+1}$ or $x_i < x_{i+1}$ and $x_j > x_{j+1}$. Moreover, such cases cover all self-intersections. So, the number of self-intersections cannot exceed number of pairs (i, j) with the property

$$(*) \quad i \equiv j \pmod 3, \quad \text{and} \quad (x_i < x_{i+1} \text{ and } x_j > x_{j+1}) \quad \text{or} \quad (x_i > x_{i+1} \text{ and } x_j < x_{j+1}).$$



Grouping the indices $1, 2, \ldots, 3n$, by remainders modulo 3, we have n indices in each residue class. Altogether there are $3\binom{n}{2}$ index pairs (i, j) with $i \equiv j \pmod 3$. We will show that for every integer k with $1 \le k < \frac{n}{2}$, there is some index i such that the pair $(i, i + 6k)$ does not satisfy $(*)$. This is already $\left[\frac{n-1}{2}\right]$ pair; this will prove that there are at most

$$3\binom{n}{2} - \left[\frac{n-1}{2}\right] \ge \frac{3}{2}n^2 - 2n + 1$$

self-intersections.

Without loss of generality we may assume that $x_{3n} = x_0$ is the smallest among $x_1, \ldots, x_{3n}$. Suppose that all of the pairs

$$(-6k, 0), \quad (-6k + 1, 1), \quad (-6k + 2, 2), \quad \ldots, \quad (-1, 6k - 1), \quad (0, 6k) \qquad (**)$$

satisfy $(*)$. Since x_0 is minimal, we have $x_{-6k} > x_0$. The pair $(-6k, 0)$ satisfies $(*)$, so $x_{-6k+1} < x_1$. Then we can see that $x_{-6k+2} > x_2$, and so on; finally we get $x_0 > x_{6k}$. But this contradicts the minimality of x_0. Therefore, there is a pair in $(**)$ that does not satisfy $(*)$.

Remark. The bound $3\binom{n}{2} - \left[\frac{n-1}{2}\right] = \left[\frac{3}{2}n^2 - 2n + 1\right]$ is sharp.