IMC2010, Blagoevgrad, Bulgaria
Day 2, July 27, 2010

Problem 1. (a) A sequence $x_1, x_2, \dots$ of real numbers satisfies

$$x_{n+1} = x_n \cos x_n \quad \text{for all} \quad n \ge 1\,.$$

Does it follow that this sequence converges for all initial values x_1?

(b) A sequence $y_1, y_2, \dots$ of real numbers satisfies

$$y_{n+1} = y_n \sin y_n \quad \text{for all} \quad n \ge 1\,.$$

Does it follow that this sequence converges for all initial values y_1?

Solution 1. (a) NO. For example, for $x_1 = \pi$ we have $x_n = (-1)^{n-1}\pi$, and the sequence is divergent.

(b) YES. Notice that $|y_n|$ is nonincreasing and hence converges to some number $a \ge 0$.

If $a = 0$, then $\lim y_n = 0$ and we are done. If $a > 0$, then $a = \lim |y_{n+1}| = \lim |y_n \sin y_n| = a\cdot|\sin a|$, so $\sin a = \pm 1$ and $a = (k+\frac{1}{2})\pi$ for some nonnegative integer k.

Since the sequence $|y_n|$ is nonincreasing, there exists an index n_0 such that $(k+\frac{1}{2})\pi \le |y_n| < (k+1)\pi$ for all $n > n_0$. Then all the numbers $y_{n_0+1}, y_{n_0+2}, \dots$ lie in the union of the intervals $\big[(k+\frac{1}{2})\pi, (k+1)\pi\big)$ and $\big(-(k+1)\pi, -(k+\frac{1}{2})\pi\big]$.

Depending on the parity of k, in one of the intervals $\big[(k+\frac{1}{2})\pi, (k+1)\pi\big)$ and $\big(-(k+1)\pi, -(k+\frac{1}{2})\pi\big]$ the values of the sine function is positive; denote this interval by I_+. In the other interval the sine function is negative; denote this interval by I_-. If $y_n \in I_-$ for some $n > n_0$ then y_n and $y_{n+1} = y_n \sin y_n$ have opposite signs, so $y_{n+1} \in I_+$. On the other hand, if If $y_n \in I_+$ for some $n > n_0$ then y_n and y_{n+1} have the same sign, so $y_{n+1} \in I_+$. In both cases, $y_{n+1} \in I_+$.

We obtained that the numbers $y_{n_0+2}, y_{n_0+3}, \dots$ lie in I_+, so they have the same sign. Since $|y_n|$ is convergent, this implies that the sequence (y_n) is convergent as well.

Solution 2 for part (b). Similarly to the first solution, $|y_n| \to a$ for some real number a.

Notice that $t \cdot \sin t = (-t)\sin(-t) = |t| \sin |t|$ for all real t, hence $y_{n+1} = |y_n| \sin |y_n|$ for all $n \ge 2$. Since the function $t \mapsto t \sin t$ is continuous, $y_{n+1} = |y_n| \sin |y_n| \to |a| \sin |a| = a$.

Problem 2. Let $a_0, a_1, \dots, a_n$ be positive real numbers such that $a_{k+1} - a_k \ge 1$ for all $k = 0, 1, \dots, n-1$. Prove that

$$1 + \frac{1}{a_0}\left(1 + \frac{1}{a_1 - a_0}\right) \cdots \left(1 + \frac{1}{a_n - a_0}\right) \le \left(1 + \frac{1}{a_0}\right)\left(1 + \frac{1}{a_1}\right) \cdots \left(1 + \frac{1}{a_n}\right).$$

Solution. Apply induction on n. Considering the empty product as 1, we have equality for $n = 0$.

Now assume that the statement is true for some n and prove it for $n+1$. For $n+1$, the statement can be written as the sum of the inequalities

$$1 + \frac{1}{a_0}\left(1 + \frac{1}{a_1 - a_0}\right) \cdots \left(1 + \frac{1}{a_n - a_0}\right) \le \left(1 + \frac{1}{a_0}\right) \cdots \left(1 + \frac{1}{a_n}\right)$$

(which is the induction hypothesis) and

$$\frac{1}{a_0}\left(1 + \frac{1}{a_1 - a_0}\right) \cdots \left(1 + \frac{1}{a_n - a_0}\right) \cdot \frac{1}{a_{n+1} - a_0} \le \left(1 + \frac{1}{a_0}\right) \cdots \left(1 + \frac{1}{a_n}\right) \cdot \frac{1}{a_{n+1}}. \qquad (1)$$

Hence, to complete the solution it is sufficient to prove (1).

To prove (1), apply a second induction. For $n=0$, we have to verify

$$\frac{1}{a_0}\cdot\frac{1}{a_1-a_0}\le\left(1+\frac{1}{a_0}\right)\frac{1}{a_1}.$$

Multiplying by $a_0a_1(a_1-a_0)$, this is equivalent with

$$\begin{aligned} a_1 &\le (a_0+1)(a_1-a_0)\\ a_0 &\le a_0a_1-a_0^2\\ 1 &\le a_1-a_0. \end{aligned}$$

For the induction step it is sufficient that

$$\left(1+\frac{1}{a_{n+1}-a_0}\right)\cdot\frac{a_{n+1}-a_0}{a_{n+2}-a_0}\le\left(1+\frac{1}{a_{n+1}}\right)\cdot\frac{a_{n+1}}{a_{n+2}}.$$

Multiplying by $(a_{n+2}-a_0)a_{n+2}$,

$$\begin{aligned} (a_{n+1}-a_0+1)a_{n+2} &\le (a_{n+1}+1)(a_{n+2}-a_0)\\ a_0 &\le a_0a_{n+2}-a_0a_{n+1}\\ 1 &\le a_{n+2}-a_{n+1}. \end{aligned}$$

Remark 1. It is easy to check from the solution that equality holds if and only if $a_{k+1}-a_k=1$ for all k.

Remark 2. The statement of the problem is a direct corollary of the identity

$$1+\sum_{i=0}^{n}\left(\frac{1}{x_i}\prod_{j\ne i}\left(1+\frac{1}{x_j-x_i}\right)\right)=\prod_{i=0}^{n}\left(1+\frac{1}{x_i}\right).$$

Problem 3. Denote by S_n the group of permutations of the sequence $(1,2,\dots,n)$. Suppose that G is a subgroup of S_n, such that for every $\pi\in G\setminus\{e\}$ there exists a unique $k\in\{1,2,\dots,n\}$ for which $\pi(k)=k$. (Here e is the unit element in the group S_n.) Show that this k is the same for all $\pi\in G\setminus\{e\}$.

Solution. Let us consider the action of G on the set $X=\{1,\dots,n\}$. Let

$$G_x=\{\,g\in G\colon\ g(x)=x\}\quad\text{and}\quad Gx=\{\,g(x)\colon\ g\in G\}$$

be the stabilizer and the orbit of $x\in X$ under this action, respectively. The condition of the problem states that

$$G=\bigcup_{x\in X}G_x \tag{1}$$

and

$$G_x\cap G_y=\{e\}\quad\text{for all}\quad x\ne y. \tag{2}$$

We need to prove that $G_x=G$ for some $x\in X$.

Let $Gx_1,\dots,Gx_k$ be the distinct orbits of the action of G. Then one can write (1) as

$$G=\bigcup_{i=1}^{k}\bigcup_{y\in Gx_i}G_y. \tag{3}$$

It is well known that

$$|Gx| = \frac{|G|}{|G_x|}. \tag{4}$$

Also note that if $y \in Gx$ then $Gy = Gx$ and thus $|Gy| = |Gx|$. Therefore,

$$|G_x| = \frac{|G|}{|Gx|} = \frac{|G|}{|Gy|} = |G_y| \quad \text{for all} \quad y \in Gx. \tag{5}$$

Combining (3), (2), (4) and (5) we get

$$|G| - 1 = |G \setminus \{e\}| = \left| \bigcup_{i=1}^{k} \bigcup_{y \in Gx_i} G_y \setminus \{e\} \right| = \sum_{i=1}^{k} \frac{|G|}{|G_{x_i}|}(|G_{x_i}| - 1),$$

hence

$$1 - \frac{1}{|G|} = \sum_{i=1}^{k} \left(1 - \frac{1}{|G_{x_i}|}\right). \tag{6}$$

If for some $i, j \in \{1, \dots, k\}$ $|G_{x_i}|, |G_{x_i}| \geq 2$ then

$$\sum_{i=1}^{k} \left(1 - \frac{1}{|G_{x_i}|}\right) \geq \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right) = 1 > 1 - \frac{1}{|G|}$$

which contradicts with (6), thus we can assume that

$$|G_{x_1}| = \ldots = |G_{x_{k-1}}| = 1.$$

Then from (6) we get $|G_{x_k}| = |G|$, hence $G_{x_k} = G$.

Problem 4. Let A be a symmetric $m \times m$ matrix over the two-element field all of whose diagonal entries are zero. Prove that for every positive integer n each column of the matrix A^n has a zero entry.

Solution. Denote by e_k $(1 \leq k \leq m)$ the m-dimensional vector over F_2, whose k-th entry is 1 and all the other elements are 0. Furthermore, let u be the vector whose all entries are 1. The k-th column of A^n is $A^n e_k$. So the statement can be written as $A^n e_k \neq u$ for all $1 \leq k \leq m$ and all $n \geq 1$.

For every pair of vectors $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$, define the bilinear form $(x, y) = x^T y = x_1 y_1 + \ldots + x_m y_m$. The product (x, y) has all basic properties of scalar products (except the property that $(x, x) = 0$ implies $x = 0$). Moreover, we have $(x, x) = (x, u)$ for every vector $x \in F_2^m$.

It is also easy to check that $(w, Aw) = w^T A w = 0$ for all vectors w, since A is symmetric and its diagonal elements are 0.

Lemma. Suppose that $v \in F_2^m$ a vector such that $A^n v = u$ for some $n \geq 1$. Then $(v, v) = 0$.

Proof. Apply induction on n. For odd values of n we prove the lemma directly. Let $n = 2k + 1$ and $w = A^k v$. Then

$$(v, v) = (v, u) = (v, A^n v) = v^T A^n v = v^T A^{2k+1} v = (A^k v, A^{k+1} v) = (w, Aw) = 0.$$

Now suppose that n is even, $n = 2k$, and the lemma is true for all smaller values of n. Let $w = A^k v$; then $A^k w = A^n v = u$ and thus we have $(w, w) = 0$ by the induction hypothesis. Hence,

$$(v, v) = (v, u) = v^T A^n v = v^T A^{2k} v = (A^k v)^T (A^k v) = (A^k v, A^k v) = (w, w) = 0.$$

The lemma is proved.

Now suppose that $A^n e_k = u$ for some $1 \le k \le m$ and positive integer n. By the Lemma, we should have $(e_k, e_k) = 0$. But this is impossible because $(e_k, e_k) = 1 \ne 0$.

Problem 5. Suppose that for a function $f \colon \mathbb{R} \to \mathbb{R}$ and real numbers $a < b$ one has $f(x) = 0$ for all $x \in (a,b)$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$ if

$$\sum_{k=0}^{p-1} f\left(y + \frac{k}{p}\right) = 0$$

for every prime number p and every real number y.

Solution. Let $N > 1$ be some integer to be defined later, and consider set of real polynomials

$$\mathcal{J}_N = \left\{ c_0 + c_1 x + \ldots + c_n x^n \in \mathbb{R}[x] \;\middle|\; \forall x \in \mathbb{R} \;\; \sum_{k=0}^{n} c_k f\Big(x + \frac{k}{N}\Big) = 0 \right\}.$$

Notice that $0 \in \mathcal{J}_N$, any linear combinations of any elements in $\mathcal{J}_N$ is in $\mathcal{J}_N$, and for every $P(x) \in \mathcal{J}_N$ we have $xP(x) \in \mathcal{J}_N$. Hence, $\mathcal{J}_N$ is an ideal of the ring $\mathbb{R}[x]$.

By the problem's conditions, for every prime divisors of N we have $\dfrac{x^N - 1}{x^{N/p} - 1} \in \mathcal{J}_N$. Since $\mathbb{R}[x]$ is a principal ideal domain (due to the Euclidean algorithm), the greatest common divisor of these polynomials is an element of $\mathcal{J}_N$. The complex roots of the polynomial $\dfrac{x^N - 1}{x^{N/p} - 1}$ are those Nth roots of unity whose order does not divide N/p. The roots of the greatest common divisor is the intersection of such sets; it can be seen that the intersection consist of the primitive Nth roots of unity. Therefore,

$$\gcd \left\{ \frac{x^N - 1}{x^{N/p} - 1} \;\middle|\; p \big| N \right\} = \Phi_N(x)$$

is the Nth cyclotomic polynomial. So $\Phi_N \in \mathcal{J}_N$, which polynomial has degree $\varphi(N)$.

Now choose N in such a way that $\dfrac{\varphi(N)}{N} < b - a$. It is well-known that $\liminf\limits_{N\to\infty} \dfrac{\varphi(N)}{N} = 0$, so there exists such a value for N. Let $\Phi_N(x) = a_0 + a_1 x + \ldots + a_{\varphi(N)} x^{\varphi(N)}$ where $a_{\varphi(N)} = 1$ and $|a_0| = 1$. Then, by the definition of $\mathcal{J}_N$, we have $\sum\limits_{k=0}^{\varphi(N)} a_k f\big(x + \frac{k}{N}\big) = 0$ for all $x \in \mathbb{R}$.

If $x \in [b, b + \frac{1}{N})$, then

$$f(x) = -\sum_{k=0}^{\varphi(N)-1} a_k f\big(x - \tfrac{\varphi(N)-k}{N}\big).$$

On the right-hand side, all numbers $x - \frac{\varphi(N)-k}{N}$ lie in (a,b). Therefore the right-hand side is zero, and $f(x) = 0$ for all $x \in [b, b + \frac{1}{N})$. It can be obtained similarly that $f(x) = 0$ for all $x \in (a - \frac{1}{N}, a]$ as well. Hence, $f = 0$ in the interval $(a - \frac{1}{N}, b + \frac{1}{N})$. Continuing in this fashion we see that f must vanish everywhere.