IMC2007, Blagoevgrad, Bulgaria

Day 2, August 6, 2007

Problem 1. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Suppose that for any $c>0$, the graph of f can be moved to the graph of cf using only a translation or a rotation. Does this imply that $f(x)=ax+b$ for some real numbers a and b ?

Solution. No. The function $f(x)=e^x$ also has this property since $ce^x=e^{x+\log c}$.

Problem 2. Let x, y, and z be integers such that $S=x^4+y^4+z^4$ is divisible by 29. Show that S is divisible by 29^4.

Solution. We claim that $29\,|\,x,y,z$. Then, $x^4+y^4+z^4$ is clearly divisible by 29^4.

Assume, to the contrary, that 29 does not divide all of the numbers x,y,z. Without loss of generality, we can suppose that $29\nmid x$. Since the residue classes modulo 29 form a field, there is some $w\in\mathbb{Z}$ such that $xw\equiv 1 \pmod{29}$. Then, $(xw)^4+(yw)^4+(zw)^4$ is also divisible by 29. So we can assume that $x\equiv 1 \pmod{29}$.

Thus, we need to show that $y^4+z^4\equiv -1 \pmod{29}$, i.e. $y^4\equiv -1-z^4 \pmod{29}$, is impossible. There are only eight fourth powers modulo 29,

$$\begin{aligned}
0 &\equiv 0^4,\\
1 &\equiv 1^4\equiv 12^4\equiv 17^4\equiv 28^4 \pmod{29},\\
7 &\equiv 8^4\equiv 9^4\equiv 20^4\equiv 21^4 \pmod{29},\\
16 &\equiv 2^4\equiv 5^4\equiv 24^4\equiv 27^4 \pmod{29},\\
20 &\equiv 6^4\equiv 14^4\equiv 15^4\equiv 23^4 \pmod{29},\\
23 &\equiv 3^4\equiv 7^4\equiv 22^4\equiv 26^4 \pmod{29},\\
24 &\equiv 4^4\equiv 10^4\equiv 19^4\equiv 25^4 \pmod{29},\\
25 &\equiv 11^4\equiv 13^4\equiv 16^4\equiv 18^4 \pmod{29}.
\end{aligned}$$

The differences $-1-z^4$ are congruent to 28, 27, 21, 12, 8, 5, 4, and 3. None of these residue classes is listed among the fourth powers.

Problem 3. Let C be a nonempty closed bounded subset of the real line and $f:C\to C$ be a nondecreasing continuous function. Show that there exists a point $p\in C$ such that $f(p)=p$.

(A set is closed if its complement is a union of open intervals. A function g is nondecreasing if $g(x)\le g(y)$ for all $x\le y$.)

Solution. Suppose $f(x)\ne x$ for all $x\in C$. Let $[a,b]$ be the smallest closed interval that contains C. Since C is closed, $a,b\in C$. By our hypothesis $f(a)>a$ and $f(b)<b$. Let $p=\sup\{x\in C:\quad f(x)>x\}$. Since C is closed and f is continuous, $f(p)\ge p$, so $f(p)>p$. For all $x>p$, $x\in C$ we have $f(x)<x$. Therefore $f\big(f(p)\big)<f(p)$ contrary to the fact that f is non-decreasing.

Problem 4. Let $n>1$ be an odd positive integer and $A=(a_{ij})_{i,j=1\ldots n}$ be the $n\times n$ matrix with

$$a_{ij}=\begin{cases}2 & \text{if } i=j\\ 1 & \text{if } i-j\equiv\pm 2 \pmod{n}\\ 0 & \text{otherwise.}\end{cases}$$

Find $\det A$.

Solution. Notice that $A = B^2$, with $b_{ij} = \begin{cases} 1 & \text{if } i-j \equiv \pm 1 \pmod{n} \\ 0 & \text{otherwise} \end{cases}$. So it is sufficient to find $\det B$.

To find $\det B$, expand the determinant with respect to the first row, and then expad both terms with respect to the first column.

$$
\begin{aligned}
\det B &= \begin{vmatrix} 0 & 1 & & & & & 1 \\ 1 & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ & & 1 & \ddots & \ddots & & \\ & & & \ddots & 0 & 1 & \\ & & & & 1 & 0 & 1 \\ 1 & & & & & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & & & & \\ & 0 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & 0 & 1 & \\ & & & 1 & 0 & 1 \\ 1 & & & & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & 0 & 1 \\ & & & & 1 & 0 \\ 1 & & & & & 1 \end{vmatrix} \\
&= - \left(\begin{vmatrix} 0 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & & & & \\ 0 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & 0 & 1 & \\ & & 1 & 0 & 1 \end{vmatrix} \right) + \left(\begin{vmatrix} 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & 1 & 0 \\ & & & & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & 1 & 0 \end{vmatrix} \right) \\
&= -(0-1) + (1-0) = 2,
\end{aligned}
$$

since the second and the third matrices are lower/upper triangular, while in the first and the fourth matrices we have $\text{row}_1 - \text{row}_3 + \text{row}_5 - \cdots \pm \text{row}_{n-2} = \bar{0}$.

So $\det B = 2$ and thus $\det A = 4$.

Problem 5. For each positive integer k, find the smallest number n_k for which there exist real $n_k \times n_k$ matrices $A_1, A_2, \ldots, A_k$ such that all of the following conditions hold:

(1) $A_1^2 = A_2^2 = \ldots = A_k^2 = 0$,

(2) $A_i A_j = A_j A_i$ for all $1 \le i, j \le k$, and

(3) $A_1 A_2 \ldots A_k \ne 0$.

Solution. The anwser is $n_k = 2^k$. In that case, the matrices can be constructed as follows: Let V be the n-dimensional real vector space with basis elements $[S]$, where S runs through all $n = 2^k$ subsets of $\{1, 2, \ldots, k\}$. Define A_i as an endomorphism of V by

$$A_i[S] = \begin{cases} 0 & \text{if } i \in S \\ [S \cup \{i\}] & \text{if } i \notin S \end{cases}$$

for all $i = 1, 2, \ldots, k$ and $S \subset \{1, 2, \ldots, k\}$. Then $A_i^2 = 0$ and $A_i A_j = A_j A_i$. Furthermore,

$$A_1 A_2 \ldots A_k [\emptyset] = [\{1, 2, \ldots, k\}],$$

and hence $A_1 A_2 \ldots A_k \ne 0$.

Now let $A_1, A_2, \ldots, A_k$ be $n \times n$ matrices satisfying the conditions of the problem; we prove that $n \ge 2^k$. Let v be a real vector satisfying $A_1 A_2 \ldots A_k v \ne 0$. Denote by $\mathcal{P}$ the set of all subsets of $\{1, 2, \ldots, k\}$. Choose a complete ordering $\prec$ on $\mathcal{P}$ with the property

$$X \prec Y \quad \Rightarrow \quad |X| \le |Y| \quad \text{for all } X, Y \in \mathcal{P}.$$

For every element $X = \{x_1, x_2, \dots, x_r\} \in \mathcal{P}$, define $A_X = A_{x_1} A_{x_2} \dots A_{x_r}$ and $v_X = A_X v$. Finally, write $\bar{X} = \{1, 2, \dots, k\} \setminus X$ for the complement of X.

Now take $X, Y \in \mathcal{P}$ with $X \not\supseteq Y$. Then $A_{\bar{X}}$ annihilates v_Y, because $X \not\supseteq Y$ implies the existence of some $y \in Y \setminus X = Y \cap \bar{X}$, and

$$A_{\bar{X}} v_Y = A_{\bar{X} \setminus \{y\}} A_y A_y v_{Y \setminus \{y\}} = 0,$$

since $A_y^2 = 0$. So, $A_{\bar{X}}$ annihilates the span of all the v_Y with $X \not\supseteq Y$. This implies that v_X does not lie in this span, because $A_{\bar{X}} v_X = v_{\{1,2,\dots,k\}} \neq 0$. Therefore, the vectors v_X (with $X \in \mathcal{P}$) are linearly independent; hence $n \geq |\mathcal{P}| = 2^k$.

Problem 6. Let $f \neq 0$ be a polynomial with real coefficients. Define the sequence $f_0, f_1, f_2, \dots$ of polynomials by $f_0 = f$ and $f_{n+1} = f_n + f_n'$ for every $n \geq 0$. Prove that there exists a number N such that for every $n \geq N$, all roots of f_n are real.

Solution. For the proof, we need the following

Lemma 1. For any polynomial g, denote by $d(g)$ the minimum distance of any two of its real zeros ($d(g) = \infty$ if g has at most one real zero). Assume that g and $g + g'$ both are of degree $k \geq 2$ and have k distinct real zeros. Then $d(g + g') \geq d(g)$.

Proof of Lemma 1: Let $x_1 < x_2 < \dots < x_k$ be the roots of g. Suppose a, b are roots of $g + g'$ satisfying $0 < b - a < d(g)$. Then, a, b cannot be roots of g, and

$$\frac{g'(a)}{g(a)} = \frac{g'(b)}{g(b)} = -1. \tag{1}$$

Since $\frac{g'}{g}$ is strictly decreasing between consecutive zeros of g, we must have $a < x_j < b$ for some j.

For all $i = 1, 2, \dots, k-1$ we have $x_{i+1} - x_i > b - a$, hence $a - x_i > b - x_{i+1}$. If $i < j$, both sides of this inequality are negative; if $i \geq j$, both sides are positive. In any case, $\frac{1}{a - x_i} < \frac{1}{b - x_{i+1}}$, and hence

$$\frac{g'(a)}{g(a)} = \sum_{i=1}^{k-1} \frac{1}{a - x_i} + \underbrace{\frac{1}{a - x_k}}_{<0} < \sum_{i=1}^{k-1} \frac{1}{b - x_{i+1}} + \underbrace{\frac{1}{b - x_1}}_{>0} = \frac{g'(b)}{g(b)}$$

This contradicts (1).

Now we turn to the proof of the stated problem. Denote by m the degree of f. We will prove by induction on m that f_n has m distinct real zeros for sufficiently large n. The cases $m = 0, 1$ are trivial; so we assume $m \geq 2$. Without loss of generality we can assume that f is monic. By induction, the result holds for f', and by ignoring the first few terms we can assume that f_n' has $m - 1$ distinct real zeros for all n. Let us denote these zeros by $x_1^{(n)} > x_2^{(n)} > \dots > x_{m-1}^{(n)}$. Then f_n has minima in $x_1^{(n)}, x_3^{(n)}, x_5^{(n)}, \dots$, and maxima in $x_2^{(n)}, x_4^{(n)}, x_6^{(n)}, \dots$. Note that in the interval $(x_{i+1}^{(n)}, x_i^{(n)})$, the function $f_{n+1}' = f_n' + f_n''$ must have a zero (this follows by applying Rolle's theorem to the function $e^x f_n'(x)$); the same is true for the interval $(-\infty, x_{m-1}^{(n)})$. Hence, in each of these $m - 1$ intervals, f_{n+1}' has *exactly* one zero. This shows that

$$x_1^{(n)} > x_1^{(n+1)} > x_2^{(n)} > x_2^{(n+1)} > x_3^{(n)} > x_3^{(n+1)} > \dots \tag{2}$$

Lemma 2. We have $\lim_{n\to\infty} f_n(x_j^{(n)}) = -\infty$ if j is odd, and $\lim\limits_{n\to\infty} f_n(x_j^{(n)}) = +\infty$ if j is even.

Lemma 2 immediately implies the result: For sufficiently large n, the values of all maxima of f_n are positive, and the values of all minima of f_n are negative; this implies that f_n has m distinct zeros.

Proof of Lemma 2: Let $d = \min\{d(f'), 1\}$; then by Lemma 1, $d(f_n') \geq d$ for all n. Define $\varepsilon = \dfrac{(m-1)d^{m-1}}{m^{m-1}}$; we will show that

$$f_{n+1}(x_j^{(n+1)}) \geq f_n(x_j^{(n)}) + \varepsilon \quad \text{for } j \text{ even.} \tag{3}$$

(The corresponding result for odd j can be shown similarly.) Do to so, write $f = f_n$, $b = x_j^{(n)}$, and choose a satisfying $d \leq b - a \leq 1$ such that f' has no zero inside (a, b). Define ξ by the relation $b - \xi = \dfrac{1}{m}(b - a)$; then $\xi \in (a, b)$. We show that $f(\xi) + f'(\xi) \geq f(b) + \varepsilon$.

Notice, that

$$\begin{aligned}
\frac{f''(\xi)}{f'(\xi)} &= \sum_{i=1}^{m-1} \frac{1}{\xi - x_i^{(n)}} \\
&= \sum_{i<j} \underbrace{\frac{1}{\xi - x_i^{(n)}}}_{<\frac{1}{\xi-a}} + \frac{1}{\xi - b} + \sum_{i>j} \underbrace{\frac{1}{\xi - x_i^{(n)}}}_{<0} \\
&< (m-1)\frac{1}{\xi - a} + \frac{1}{\xi - b} = 0.
\end{aligned}$$

The last equality holds by definition of ξ. Since f' is positive and $\dfrac{f''}{f'}$ is decreasing in (a, b), we have that f'' is negative on (ξ, b). Therefore,

$$f(b) - f(\xi) = \int_\xi^b f'(t)dt \leq \int_\xi^b f'(\xi)dt = (b - \xi)f'(\xi)$$

Hence,

$$\begin{aligned}
f(\xi) + f'(\xi) &\geq f(b) - (b - \xi)f'(\xi) + f'(\xi) \\
&= f(b) + (1 - (\xi - b))f'(\xi) \\
&= f(b) + (1 - \tfrac{1}{m}(b - a))f'(\xi) \\
&\geq f(b) + (1 - \tfrac{1}{m})f'(\xi).
\end{aligned}$$

Together with

$$f'(\xi) = |f'(\xi)| = m \prod_{i=1}^{m-1} \underbrace{|\xi - x_i^{(n)}|}_{\geq|\xi-b|} \geq m|\xi - b|^{m-1} \geq \frac{d^{m-1}}{m^{m-2}}$$

we get

$$f(\xi) + f'(\xi) \geq f(b) + \varepsilon.$$

Together with (2) this shows (3). This finishes the proof of Lemma 2.